NUMERICAL SOLUTIONS OF ODEs.

Anyone with passing acquaintance with math gets told that differential equations model a wide range of physical / biological / financial / econ. phenomenon.

But when one is asked for explicit examples, one is often told serious examples need PDEs. In fact ODEs are already rich enough. I won't discuss any of these examples but if you're interested, please look them up:

PHYSICAL PHENOMENON

- Mechanical systems
- Lorenz attractors
- Rate of reactions
- climate modeling etc.

BIOLOGICAL PHENOMENON

- Population dynamics
- Modeling of disease spread
- Hodgkin - Huxley Neuron etc.

Econ. growth model from Neoclassical growth theory

(Eg. Solow, Harrod-Domar, Ramsey-Cass-Koopmans, Mankiw-Romer-Weil ...) etc.

FINANCIAL PHENOMENON

⇝ Chapman-Kolmogorov ODEs for Markov models

⇝ $\mathbb{E}(\text{SODE}) = \text{ODE}$ (roughly!), so useful in calculation of mathematical reserves.

General Setup.

We are given the IVP

$$(E)\quad \begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

w/ $f : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R}$ (assumed sufficiently regular)

$\rightsquigarrow$ Usually one may not be able to solve (E) analytically. But we might still desire qualitative / quantitative analysis of exact solutions to (E).

(Typically, qualitative $\longleftrightarrow$ bulk, quantitative tends to be about instantaneous behavior. One goal of this area of Mathematics is to produce algorithms which produce approximations that fit the observed qualitative behavior...)

BASIC PROBLEM. Given a mesh

$$t_0 < t_1 < \dots < t_N = t_0 + T$$

of the interval $[t_0, t_0+T]$, produce $\{y_n\}_{1 \le n \le N}$ that approximates $z(t_n)$ for an exact solution z of (E).

$\boxed{\text{Notation}}$ $\quad h_n = t_{n+1} - t_n, \quad 0 \le n \le N-1$

$$h_{max} = \max_{0 \le n \le N-1} h_n$$

Idea. "Discretize" (E) and produce y_{n+1} from $y_n, \dots, y_{n-(r-1)}$.

$\rightsquigarrow$ usually called an r-step method

$\boxed{\text{GOAL}}$ Prescribe and analyze a family of 1-step methods.

EXAMPLE 1. (Euler's method)

Let z be an exact solution to (E). Then, by linear approximation:

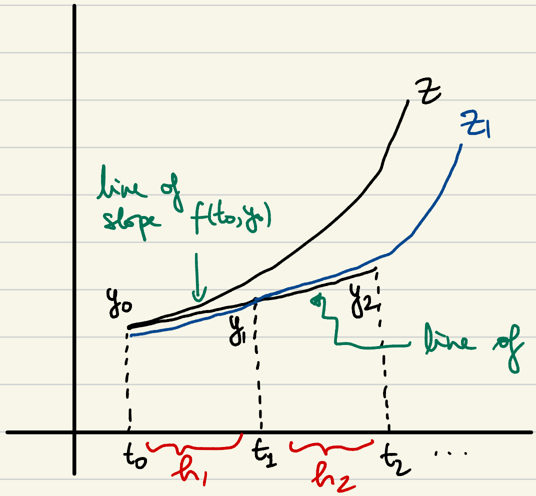
$$\begin{aligned} z(t_{n+1}) &\approx z(t_n) + h_n\, z'(t_n) \\ &= z(t_n) + h_n\, f(t_n, z(t_n)) \end{aligned}$$

This suggests the algorithm:

$\boxed{\text{ALGORITHM (Euler's Method)}}$ $(0 \leq n \leq N-1)$

$$\begin{cases} y_{n+1} = y_n + h_n f(t_n, y_n) \\ t_{n+1} = t_n + h_n. \end{cases}$$

(Note. Requires 1 evaluation of f per step.)



exact solution to
$$\begin{cases} y' = f(t, y) \\ y(t_1) = y_1 \end{cases}$$

Fig. Schematics of Euler's Method.

A new idea. Instead of approximating z by a line of slope $z'(t_n)$ on the interval $[t_n, t_{n+1}]$, we could try to use a line of slope $\underbrace{z'\left(t_n + \frac{h_n}{2}\right)}_{\text{slope of } z \text{ at the midpoint of } [t_n, t_{n+1}]}$.

$\rightsquigarrow$ This leads to the mid-point method.

EXAMPLE 2. (Midpoint Method)

As explained before, we wish to use

$$z(t_{n+1}) \approx z(t_n) + h_n z'\left(t_n + \frac{h_n}{2}\right)$$

$$= z(t_n) + h_n \cdot f\left(t_n + \frac{h_n}{2}, \underbrace{z\left(t_n + \frac{h_n}{2}\right)}\right)$$

don't know!
so approximate by Euler!

$$\approx z(t_n) + h_n \cdot f\left(t_n + \frac{h_n}{2}, z(t_n) + \frac{h_n}{2} f(t_n, z(t_n))\right)$$

ALGORITHM (Midpoint Method) $\quad 0 \le n \le N-1$

$$\begin{cases} y_{n+1/2} = y_n + \frac{h_n}{2} f(t_n, y_n) \\ y_{n+1} = y_n + h_n f\left(t_n + \frac{h_n}{2}, y_{n+1/2}\right) \\ t_{n+1} = t_n + h_n \end{cases}$$

(Note: Requires 2 evaluations of f per step)

Systematic study of one-step Methods

Notations of the introduction in force.

Def 1. A 1-step method is a method of computing the approximations $\{y_n\}_{n=1,\dots,N}$ in which y_{n+1} may be written as

$$y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n) \quad \text{for } 0 \le n < N$$

where $\Phi : [t_0, t_0+T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Rmk. In practice, the function Φ is prescribed only on a set of the form $[t_0, t_0+T] \times J \times [0, \delta]$ where $J \subseteq \mathbb{R}$ is an interval such that $[t_0, t_0+T] \times J$ is contained in the domain of definition of the ODE and $\delta > 0$ is suff. small.

To analyse 1-step methods, let us introduce various measures of error.

Def 2. (Local Error)

The local (or 1-step) error relative to an exact solution z of the ODE (E) is

$$e_n = z(t_{n+1}) - \hat{y}_{n+1} \qquad 0 \le n \le N-1$$

where $\hat{y}_{n+1}$ is the approximation produced by 1-step application of our procedure with exact solution $z(t_n)$.

Thus, $\hat{y}_{n+1} = z(t_n) + h_n \Phi(t_n, z(t_n), h_n)$ and

$$\boxed{e_n = z(t_{n+1}) - z(t_n) - h_n \Phi(t_n, z(t_n), h_n)}$$

Def. 3 (global error)

The global error relative to an exact solution z of the ODE (E) is

$$\theta_n = \max_{0 \le j \le n} |z(t_j) - y_j|$$

where $\{y_j\}_{j=1,\dots,n}$ are values produced by successive application of our procedure starting from the initial value $y_0 = z(t_0)$.

Def 4. (Consistent method)

A 1-step method Φ is said to be <u>CONSISTENT</u>

if for every exact solution z, the sum of the local errors $\sum_{0 \le n \le N-1} |e_n|$ tends to 0 as

h_{max} tends to 0.

<u>Rmk. 1.</u> This notion merely addresses the purely theoretical accumulation of local errors, and <u>does NOT</u> account for the deviation of the calculated approximation from the actual solution!!

<u>Rmk. 2.</u> An astute reader will note that we have fixed a mesh of $[t_0, t_0+T]$ in the introduction, but most analyses will produce an expression for $\sum_{0 \le n \le N} |e_n|$ only in terms of h_{max}.

So, we will require consistency independent of any specific sequence of meshes of $[t_0, t_0+T]$ along which $h_{max} \to 0$.

Def. 5. (Convergent Method)

A 1-step method Φ is said to be <u>CONVERGENT</u> if for every exact solution z, the sequence $\{y_n\}_{n \ge 0}$ produced using Φ, i.e.,

$$y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n)$$

satisfies

$$\max_{0 \le n \le N} |y_n - z(t_n)| \to 0 \quad \text{as } h_{max} \to 0.$$

In practice, an applied mathematician must also account for the limitations of computing, e.g. round-off errors.

So, let $\{\tilde{y}_n\}_{0\le n\le N}$ be the approximations computed using the 1-step method prescribed by Φ. Thus:

whereas
$$y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n) \qquad (\dagger)$$
one really computes
$$\tilde{y}_{n+1} = \tilde{y}_n + h_n \Phi(t_n, \tilde{y}_n, h_n) + \varepsilon_n \qquad (\ddagger)$$

ε_n: machine error in the n-th step

We now introduce a notion of "goodness" for a method that accounts for this:

Def. 6. (Stable method)

A 1-step method Φ is said to be <u>STABLE</u> if there exists a constant S such that

$$\max_{0\le n\le N} |\tilde{y}_n - y_n| = S\left(|\tilde{y}_0 - y_0| + \sum_{0\le n<N} |\varepsilon_n|\right)$$

for all sequences $\{\tilde{y}_n\}$ and $\{y_n\}$ given by $(\dagger)$ and $(\ddagger)$

i.e. the successive errors in $\tilde{y}_n$ as compared to the value y_n, viz. that prescribed by our method, stay under control if our initial error and the machine errors stay under control.

<u>Rmk.</u> The optimal S, if it exists, is called the constant of stability.

<u>HOPE</u>. If a method is consistent and stable, then it is convergent.

<u>Lemma</u>. <u>HOPE</u> ok.

<u>Proof</u>. Note that, by definition of local error,

$$z(t_{n+1}) = z(t_n) + h_n \Phi(t_n, y_n, h_n) + e_n$$

If we assume our 1-step method is stable, the defi. w/ $\tilde{y}_{n+1} = z(t_{n+1})$ gives:

$$\max_{0 \le n \le N} |y_n - z(t_n)| \le S' \Big(\underbrace{|y_0 - z(t_0)|}_{0} + \underbrace{\sum_{0 \le n < N} |e_n|}_{0} \Big)$$

by assumption

by consistency.

$\rightsquigarrow$ Convergent. ∎

Exercises

1. Calculate the local and global error in Euler's Method and the Midpoint Method.

2. For a triple of real numbers (α, β, γ) with $0 \leq \alpha, \beta, \gamma \leq 1$, consider the 1-step method

TRY, (TRY AGAIN AFTER LEC 2)

$$y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n)$$

where

$$\Phi(t, y, h) = \alpha f(t, y) + \beta f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right) + \gamma f(t+h, y + h f(t, h))$$

(a) Assume that $f(t, y)$ is Lipschitz in y w/ Lipschitz constant k, i.e.

$$|f(t, y_1) - f(t, y_2)| < k |y_1 - y_2|$$

$\forall (t, y_1), (t, y_2) \in [t_0, t_0+T] \times \mathbb{R}$. Assume also that f is C^∞ on $[t_0, t_0+T] \times \mathbb{R}$.

For what triples (α, β, γ) is Φ stable?

(b) For what triples (α, β, γ) is Φ consistent! convergent?