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End-semester Examination for MT3164: Numerical Analysis
Date: 25^{th} Nov 2025 (Tuesday) Time: 03:00 pm – 05:00 pm (2 hours)
Number of questions: 5; Maximum number of points: 40

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1. (a) 1. For the pair (x_n, α_n), is it true that $x_n = \mathcal{O}(\alpha_n)$ as $n \to \infty$? $\boxed{4}$

   (i) $x_n = 5n^2 + 9n^3 + 1, \quad \alpha_n = n^2$ (iii) $x_n = \sqrt{n+3}, \quad \alpha_n = 1$
   (ii) $x_n = 5n^3 + 9n^3 + 1, \quad \alpha_n = 1$ (iv) $x_n = \sqrt{n+3}, \quad \alpha_n = 1/n$

   2. Let u, v be continous real-valued functions on an interval $[a, b]$, and suppose that $v \geq 0$. Then there exists a point η in $[a, b]$ such that

$$\int_a^b u(x)v(x)dx = u(\eta) \int_a^b v(x)dx.$$

   (b) Consider two $n \times 1$ vectors b and $\tilde{b}$. Suppose x and $\tilde{x}$ satisfies $Ax = b$ and $A\tilde{x} = \tilde{b}$, respectively, where A is $n \times n$ invertible matrix. Derive an expression for relative difference in x and $\tilde{x}$ in terms of $b, \tilde{b}$ and the condition number of A. $\boxed{4}$

   Prove that the condition number of A is greater than or equal to 1.

2. (a) Consider a $m \times n$ real matrix A. Suppose we denote column subspace, null subspace, row subspace, and left null subspace of A by $\mathcal{C}(A), \mathcal{N}(A), \mathcal{C}(A^T)$ and $\mathcal{N}(A^T)$, respectively. Formally define these subspaces and prove that the orthogonal complement of the column space of A is the left null space of A. $\boxed{4}$

   (b) Describe the power method to compute dominent eigenvalue and the corresponding eigenvector of matrix A. Specify the assumptions on matrix A in which case the method converges and prove that the method indeed converges. $\boxed{4}$

3. (a) Define SVD and pseudoinverse of a $m \times n$ matrix A. Prove that the minimal solution of the equation $Ax = b$ is given by the speudoinverse $x = A^+b$. Here, the minimum solution of $Ax = b$ is a vector $x \in \mathbb{C}^n$ such that the value of $\|Ax - b\|_2$ is minimized. $\boxed{4}$

   (b) Derive Newton-Cotes formula for $\int_0^1 f(x)dx$ based on nodes $0, 1/3, 2/3$ and 1. $\boxed{4}$

4. (a) Describe the cubic spline interpolation method and list all the constraints used to determine the coefficients introduced in it. $\boxed{4}$

   (b) 1. Describe Newton form of the interpolation polynomial. $\boxed{4}$

   2. Describe how to use Vandermonde matrix for polynomial interpolation.

5. (a) 1. Derive Newton-Raphson method for computing the fifth root of any positive real number. $\boxed{4}$

   2. Desribe bisection method. Find the number of steps required to gaurantee that the error is bounded by ϵ given that method is applied in range $[a_0, b_0]$.

   (b) Derive the second-order Runge–Kutta method starting from the Taylor series expansion of the exact solution. Show how the constants are chosen so that the method achieves order 2. You may assume that the method involves just 2 stages. $\boxed{4}$