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Mid-Term Exam for MT3444: Combinatorial Optimization

Date: 23rd Feb 2026 (Monday) Time: 3:00pm – 05:00pm (2 hours)
Number of questions: 4; Maximum number of points: 30

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1. (a) Let $G = (V, E)$ be a graph and let $M \subseteq E$ be a *maximal* matching in G. Prove that the set $V(M) := \{v \in V : v \text{ is an endpoint of some edge in } M\}$ is a vertex cover of G. Further, prove that $|V(M)| \leq 2\,\tau(G)$, where $\tau(G)$ denotes the size of a minimum vertex cover of G. [3]

   (b) Consider the linear program max $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$. Describe how the Simplex method can be used to determine whether the above linear program is feasible. [3]

2. (a) A student allocates study time between Mathematics and Physics. Each hour devoted to Mathematics increases the expected score by 5 points, and each hour devoted to Physics increases the expected score by 4 points. The student has at most 10 hours available for study. At least 2 hours must be devoted to each subject, and at most 6 hours may be devoted to any one subject. [3]

   1. Formulate a linear programming model that maximizes the total expected score improvement.
   2. Write explicitly the dual of your linear program.

   (b) Consider the integer linear program (P) max $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \in \mathbb{Z}^n$. [3]

   Present non-trivial conditions under which the following statement holds: *If (P) has an optimal solution, then its optimal value equals the optimal value of its linear programming relaxation.*

3. (a) Consider the linear program max $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be feasible solutions. [4]

   Prove that $\mathbf{u}, \mathbf{v}$ are optimal solutions if and only if $\frac{\mathbf{u}+\mathbf{v}}{2}$ is an optimal solution.

   (b) A linear programming problem can be: infeasible, feasible and bounded, or feasible and unbounded. Consider the primal problem max $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and its dual. [4]

   1. (1pt) State formally the Weak Duality Theorem.
   2. (1pt) State formally the Strong Duality Theorem.
   3. (2pts) Construct a 3×3 table describing all possible combinations of statuses (infeasible, feasible and bounded, feasible and unbounded) for the primal and the dual. Clearly indicate in your table the entries ruled out by Weak Duality and those characterized by Strong Duality, and justify why.

4. (a) Consider the linear program max $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, and the Simplex tableau $\mathcal{T}(B)$ corresponding to a basis B: [5]

$$\begin{array}{rcl} \mathbf{x}_B & = & \mathbf{p} + Q\mathbf{x}_N \\ \hline z & = & z_0 + \mathbf{r}^\top \mathbf{x}_N \end{array}$$

Answer the following:

1. (1pt) Specify necessary and sufficient conditions under which:
   (a) the basic feasible solution corresponding to B is optimal;
   (b) a non-basic variable can enter the basis.
2. (1pt) What is a degenerate pivot step?
3. (1pt) State Bland's pivot rule precisely.
4. (2pts) Prove that if the Simplex method cycles, then all bases encountered during the cycle correspond to the same basic feasible solution.

(b) State and prove the geometric form of Farkas' Lemma. Formally define all concepts used in your proof. [5]

You may assume the following lemma:

Lemma 1. *Let $C \subseteq \mathbb{R}^m$ be a convex cone generated by finitely many vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. If $\boldsymbol{b} \notin C$, then there exists a point $\boldsymbol{z} \in C$ that minimizes the Euclidean distance to $\boldsymbol{b}$.*