Applications of the Max-Flow Min-Cut Theorem

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We derive two classical min–max theorems by constructing a flow network and interpreting its integral flows and cuts as combinatorial objects.

Preliminaries

A *flow network* is a directed graph $D = (V, A)$ with source s, sink t, and nonnegative arc capacities c. For $S \subseteq V$ with $s \in S$ and $t \notin S$, its cut capacity is

$$\operatorname{cap}(S, V \setminus S) = \sum_{\substack{xy \in A \\ x \in S,\ y \notin S}} c(xy).$$

The *Max-Flow Min-Cut Theorem* states that

$$\max\{|f| : f \text{ is an } s\text{–}t \text{ flow}\} = \min_{\substack{s \in S \\ t \notin S}} \operatorname{cap}(S, V \setminus S).$$

We also use *flow integrality*: integer capacities admit an integer-valued maximum flow.

1 König's theorem

Theorem 1.1 (König)**.** *For every bipartite graph $G = (A \cup B, E)$, the maximum matching size equals the minimum vertex-cover size: $\nu(G) = \tau(G)$.*

Network construction

Add a source s and a sink t. The network has arcs

$$s \longrightarrow a \quad (a \in A), \qquad a \longrightarrow b \quad (ab \in E), \qquad b \longrightarrow t \quad (b \in B),$$

with capacities $1, M, 1$, respectively, where $M = |A| + |B| + 1$.

During the class, we went with the flow of the discussion and worked with capacities of arcs ab as 1. However, taking these capacities as M will make this proof easier, and it will also connect it with the next proof. – Prafull

Flows correspond to matchings

Every matching F of size k determines an integral flow of value k: send one unit along $s \to a \to b \to t$ for each $ab \in F$. The unit capacities incident with s and t are respected because matching edges have disjoint endpoints.

Conversely, take an integral maximum flow. Each $s \to a$ and $b \to t$ arc carries at most one unit. Flow conservation implies that every $a \in A$ sends at most one unit through the A–B arcs and every $b \in B$ receives at most one. Thus, the A–B arcs carrying one unit form a matching, and their number equals the flow value. Therefore

$$\max |f| = \nu(G).$$

Cuts correspond to vertex covers

Let C be a vertex cover of G, and define

$$S = \{s\} \cup (A \setminus C) \cup (B \cap C).$$

There is no edge from $A \setminus C$ to $B \setminus C$, since C covers every edge. Hence no capacity-M arc goes from S to its complement. The cut contains precisely the arcs $s \to a$ with $a \in A \cap C$ and $b \to t$ with $b \in B \cap C$, so

$$\operatorname{cap}(S, V \setminus S) = |A \cap C| + |B \cap C| = |C|.$$

This proves $\min \operatorname{cap} \le \tau(G)$.

Conversely, fix a minimum cut $(S, T = \bar{S})$. The cut with source side $\{s\}$ has capacity $|A| < M$, so no capacity-M arc can cross a minimum cut. Set

$$C = (A \cap T) \cup (B \cap S).$$

If some edge $ab \in E$ had neither endpoint in C, then $a \in S$ and $b \in T$, making its capacity-M arc cross the cut—a contradiction. Thus C is a vertex cover. As only unit-capacity arcs cross,

$$\operatorname{cap}(S, T) = |A \cap T| + |B \cap S| = |C| \ge \tau(G).$$

Therefore $\min \operatorname{cap} = \tau(G)$.

Putting it together

Max-Flow Min-Cut now yields

$$\nu(G) = \max |f| = \min \operatorname{cap} = \tau(G). \qquad \square$$

2 Hall's marriage theorem

Theorem 2.1 (Hall)**.** *A bipartite graph $G = (A \cup B, E)$ has a matching saturating A if and only if*

$$|N(X)| \ge |X| \qquad \text{for every } X \subseteq A.$$

Network and target flow value

Use the same network as for König's theorem: unit capacities on $s \to A$ and $B \to t$, and capacity $M = |A| + |B| + 1$ on each $A \to B$ arc. By the integral flow–matching correspondence, A can be saturated exactly when the maximum flow has value $|A|$. Since the total capacity leaving s is $|A|$, it remains to determine when every cut has capacity at least $|A|$.

Sufficiency of Hall's condition

Assume $|N(X)| \ge |X|$ for all $X \subseteq A$. Let (S, T) be an s–t cut and put

$$X = A \cap S, \qquad Y = B \cap S.$$

If some edge joins X to $B \setminus Y$, its capacity-M arc crosses the cut, so $\operatorname{cap}(S, T) \ge M > |A|$. Otherwise $N(X) \subseteq Y$, and the only forward crossing arcs are $s \to (A \setminus X)$ and $Y \to t$. Consequently

$$\operatorname{cap}(S, T) = |A \setminus X| + |Y| \ge |A| - |X| + |N(X)| \ge |A|.$$

Every cut has capacity at least $|A|$, so Max-Flow Min-Cut gives a flow of value $|A|$. By integrality it yields a matching saturating A.

Necessity of Hall's condition

Suppose there is some $X \subseteq A$ with $|N(X)| < |X|$. Take the cut with source side

$$S = \{s\} \cup X \cup N(X).$$

No capacity-M arc leaves S, since all neighbours of X lie in $N(X)$. Its capacity is

$$\text{cap}(S, V \setminus S) = |A \setminus X| + |N(X)| = |A| - |X| + |N(X)| < |A|.$$

Putting it together

By Max-Flow Min-Cut, no flow of value $|A|$ exists, and hence no matching saturates A. This proves Hall's theorem. $\square$