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Template Answers for Quiz 2 for MT 3154 Graph Theory

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1. (a) **Prove that every planar graph can be vertex colored using at most six colors.** **[3 pts]**

   [1 pt] Every planar graph has a vertex of degree at most 5.
   Indeed, for a simple planar graph G with $n \geq 3$ vertices and m edges, Euler's formula implies $m \leq 3n - 6$.
   Hence $\sum_{v \in V(G)} d(v) = 2m \leq 6n - 12 < 6n$, so the average degree is strictly less than 6. Therefore some vertex has degree at most 5.

   [1 pt] We now use induction on $|V(G)|$. The claim is immediate for graphs with at most six vertices.
   Let v be a vertex with $d(v) \leq 5$. The graph $G - v$ is planar, so by induction it has a proper vertex coloring using at most six colors.

   [1 pt] At most five colors occur on the neighbours of v. Hence among the six available colors there is at least one color not used by any neighbour of v. Assign this color to v. This extends the coloring of $G - v$ to a proper 6-coloring of G.
   Thus every planar graph is vertex colorable with at most six colors.

   (b) **We say a graph is *Hamiltonian graph* if there exists a cycle that contains all the vertices of the graph. Prove that if G is a 3-regular Hamiltonian graph then $\chi'(G) = 3$.** **[5 pts]**

   [1 pt] Let C be a Hamiltonian cycle of G.
   Every vertex has two incident edges in C, and G is 3-regular. Hence every vertex is incident with exactly one edge of $M := E(G) \setminus E(C)$. Therefore M is a perfect matching.

   [1 pt] The number of vertices of G is even. Indeed, by the Handshaking Lemma, $3|V(G)| = \sum_{v \in V(G)} d(v) = 2|E(G)|$, so $|V(G)|$ is even.
   Hence the Hamiltonian cycle C has even length.

   [1 pt] Color the edges of C alternately with colors 1 and 2. Since C has even length, this gives a proper edge coloring of C.

   [1 pt] Color every edge of the perfect matching M with color 3. Since M is a matching, no two edges of color 3 are incident.
   At every vertex, the two cycle edges have colors 1 and 2, and the remaining edge has color 3. Thus this is a proper 3-edge-coloring of G, and $\chi'(G) \leq 3$.

   [1 pt] In any proper edge coloring, the three edges incident with a vertex of degree 3 must receive different colors. Hence $\chi'(G) \geq \Delta(G) = 3$. Combining the two inequalities gives $\chi'(G) = 3$.

2. (a) **Describe the *greedy algorithm* for vertex coloring of a graph G and prove that $\Delta(G)+1$ are always sufficient.** **[2 pts]**

[1 pt] Fix an ordering $v_1, v_2, \ldots, v_n$ of the vertices.
The greedy coloring algorithm considers the vertices in this order. When v_i is considered, assign to v_i the smallest available color that has not already been assigned to one of its colored neighbours.

[1 pt] When v_i is colored, it has at most $d(v_i) \leq \Delta(G)$ already colored neighbours. Therefore at most $\Delta(G)$ colors are forbidden at v_i. Among $1, 2, \ldots, \Delta(G)+1$ there is consequently always at least one available color. Thus greedy coloring always uses at most $\Delta(G)+1$ colors.

(b) **Prove that for every bipartite graph G, we have $\chi'(G) = \Delta(G)$.** **[5 pts]**

[1 pt] Clearly, $\chi'(G) \geq \Delta := \Delta(G)$, since all edges incident with a vertex of degree Δ must receive different colors.
It remains to prove that G has a proper edge coloring with Δ colors.
We prove this by induction on $|E(G)|$. The assertion is immediate when $|E(G)| = 0$.
Now suppose $|E(G)| \geq 1$, and assume the result holds for bipartite graphs with fewer edges. Choose an edge $xy \in E(G)$. By induction, $G - xy$ has a proper Δ-edge-coloring.

[1 pt] Since both x and y have degree at most $\Delta - 1$ in $G - xy$, there exist colors $\alpha, \beta \in \{1, \ldots, \Delta\}$ such that x is incident with no α-edge and y is incident with no β-edge.
If $\alpha = \beta$, we simply color xy with this color and are done.
Thus assume $\alpha \neq \beta$.
If x is incident with no β-edge either, we can color xy with β. Hence we may assume that x is incident with a β-edge.

[1 pt] Starting at x, extend this β-edge to a maximal walk W whose edges are colored alternately $\beta, \alpha, \beta, \alpha, \ldots$.
Because the edge coloring is proper, every vertex has at most one incident α-edge and at most one incident β-edge.
Hence W cannot repeat a vertex; otherwise its repeated portion would form an even α-β cycle and the walk could not enter or leave that vertex as required. Thus W is a path.

[1 pt] We claim that W does not contain y.
If W contained y, then since y has no incident β-edge, the path W would have to end at y with an α-edge. Since W begins with a β-edge, W would then have even length.
Consequently, $W + xy$ would be an odd cycle in G, contradicting the fact that G is bipartite.

[1 pt] Now interchange the colors α and β on every edge of W.
The coloring remains proper. By the choice of α and the maximality of W, after this interchange x is incident with no β-edge. Since $y \notin V(W)$, y is still incident with no β-edge.
We can therefore color xy with β. This gives a proper Δ-edge-coloring of G. Hence $\chi'(G) \leq \Delta$.

3. (a) **Construct a graph which is neither a clique nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(G)+1$ colors. Draw the graph and specify the ordering.** **[4 pts]**

There are various examples for this. – Prafull

[3 pt] For the graph. Example is path $P_4 = \langle a - b - c - d \rangle$.

[1 pt] Use the vertex ordering a, d, b, c.
If your ordering does not generate $\Delta + 1$ colors then you will not get any points. – Prafull

(b) **Latin square completion as a graph problem.** **[5 pts]**

[2 pt] Construct the complete bipartite graph $K_{n,n}$ with bipartition $R = \{r_1, \ldots, r_n\}$, and $C = \{c_1, \ldots, c_n\}$, where r_i represents row i and c_j represents column j.
Associate the cell (i, j) of the $n \times n$ array with the edge $r_i c_j$.

[1 pt] Interpret the numbers $1, 2, \ldots, n$ as edge colors. If cell (i, j) contains the number k, give the edge $r_i c_j$ color k.
Thus a partial Latin square gives a partial edge coloring of $K_{n,n}$.

[1 pt] The condition that no number occurs twice in a row says precisely that no two equally colored edges are incident with the same row vertex r_i.
Similarly, the condition that no number occurs twice in a column says that no two equally colored edges are incident with the same column vertex c_j.
Hence a Latin square is exactly a proper edge coloring of $K_{n,n}$ using the colors $\{1, \ldots, n\}$.

[1 pt] Therefore the graph-theoretic problem is: Given a partial proper edge coloring of $K_{n,n}$ with colors $\{1, \ldots, n\}$, determine whether it can be extended to a proper edge coloring of all edges of $K_{n,n}$ with these n colors, without changing any precolored edge.