Lossy Kernels for Graph Contraction Problems

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Abstract

We study some well-known graph contraction problems in the recently introduced framework of *lossy kernelization*. In classical kernelization, given an instance (I, k) of a parameterized problem, we are interested in obtaining (in polynomial time) an equivalent instance (I', k') of the same problem whose size is bounded by a polynomial in k. This notion however has a major limitation. Given an approximate solution to the instance (I', k'), we can say nothing about the original instance (I, k). To handle this issue, among others, the framework of Lossy kernelization was introduced. In this framework, for a constant α, given an instance (I, k) we obtain an instance (I', k') of the same problem such that, for every $c > 1$, any c-approximate solution to (I', k') can be turned into a $(c\alpha)$-approximate solution to the original instance (I, k) in polynomial time. Naturally, we are interested in a polynomial time algorithm for this task, and further require that $|I'| + k' = k^{\mathcal{O}(1)}$. Akin to the notion of polynomial time approximation schemes in approximation algorithms, a parameterized problem is said to admit a polynomial size approximate kernelization scheme (PSAKS) if it admits a polynomial size α-approximate kernel for every approximation parameter $\alpha > 1$. In this work, we design PSAKSs for Tree Contraction, Star Contraction, Out-Tree Contraction and Cactus Contraction problems. These problems do not admit polynomial kernels, and we show that each of them admit a PSAKS with running time $k^{f(\alpha)}|I|^{\mathcal{O}(1)}$ that returns an instance of size $k^{g(\alpha)}$ where $f(\alpha)$ and $g(\alpha)$ are constants depending on α.

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1 Introduction

Many computational problems arising from real-world problems are NP-hard, and we do not expect any efficient algorithms for solving them optimally. Preprocessing heuristics, or data reduction rules, are widely applied to reduce large instances of these problems to a smaller size before attempting to solve them. Such algorithms are often extremely effective, and provide a significant boost to the subsequent step of computing a solution to the instance. Kernelization, under the aegis of Parameterized Complexity, has been developed as a mathematical framework to study these algorithms and quantify their efficacy. In *Parameterized Complexity*, we consider instances (I, k) of parameterized a problem $\Pi \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. Typically, I is an instance of some computational problem, and k denotes the *parameter* which reflects some structural property of the instance. A common parameter is a bound on the size of an optimum solution to the problem instance. A data reduction algorithm, formally called a *Kernelization algorithm*, runs in polynomial time and reduces a given instance (I, k) of the problem to an *equivalent* instance (I', k') such that $|I'| + k' = k^{\mathcal{O}(1)}$. The instance (I', k') is called a *polynomial kernel*, and we say that the problem Π admits a polynomial kernelization (also called classical kernelization).

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Desigining kernelization algorithms for various computational problems, and investigating the associated lower bounds, is an active area of research in Computer Science. We refer the reader to [6, 9, 10] for an introduction to Parameterized Complexity and Kernelization.

The notion of polynomial kernels turns out to be a bit stringent, and it has been discovered that many problems do not admit a polynomial kernel under well-known complexity theory conjectures. On the other hand this notion turns out to be too lax as the instances (I,k) and (I',k') are not as tightly-coupled as one would like. For example, it is not possible to translate an approximate solution to the instance (I',k'), into an approxmimate solution to the original instance (I,k). Indeed, given anything but an optimal solution (or a solution of size k') to (I',k'), it is impossible to conclude anything about the original instance (I,k). These issues, among others, have led to the development of a framework for "approximation preserving kernelization" or *Lossy Kernelization*. Informally, an α-approximate kernelization algorithm ensures that given any c-approximate solution to the kernel (I',k'), it can be converted into a $c \cdot \alpha$-approximate solution to the original instance (I,k) in polynomial time. This notion was formally introduced, very recently, in [18] which shows that there are many problems without classical polynomial kernels that admit lossy polynomial kernels. Furthermore, it is likely that this notion will be very useful in practice. Many state of the art approximation algorithms are extremely sophisticated and it is infeasible to apply them to large problem instances. It is far more practical to reduce a large instance to a small kernel, then obtain a good approximate solution to this kernel, and finally transform it into an approximate solution to the original instance. In other words, lossy kernelization provides a mathematical framework for designing and analyzing preprocessing heuristics for approximation algorithms.

Let us state these notions formally. We first define a *parameterized optimization (maximization / minimization) problem*, which is the parameterized analogue of an optimization problem in the theory of approximation algorithms. A *parameterized minimization problem* is computable function $\Pi : \Sigma^* \times \mathbb{N} \times \Sigma^* \mapsto \mathbb{R} \cup \{\pm\infty\}$. The instances of Π are pairs $(I,k) \in \Sigma^* \times \mathbb{N}$ and a solution to (I,k) is simply a string $S \in \Sigma^*$ such that $|S| \leq |I|+k$. The *value* of a solution S is $\Pi(I,k,S)$. The *optimum value* of (I,k) is $\mathrm{OPT}_\Pi(I,k) = \min_{S \in \Sigma^*, |S| \leq |I|+k} \Pi(I,k,S)$. An *optimum solution* for (I,k) is a solution S such that $\Pi(I,k,S) = \mathrm{OPT}_\Pi(I,k)$. A *parameterized maximization problem* is defined in a similar way. We omit the subscript Π in the notation for optimum value if the problem under consideration is clear from context. Next we come to the notion of an *α-approximate polynomial-time preprocessing algorithm* for a parameterized optimization problem Π. It is defined as a pair of polynomial-time algorithms, called the *reduction algorithm* and the *solution lifting algorithm*, that satisfy the following properties.

- Given an instance (I,k) of Π, the reduction algorithm computes an instance (I',k') of Π.
- Given the instances (I,k) and (I',k') of Π, and a solution S' to (I',k'), the solution lifting algorithm computes a solution S to (I,k) such that $\frac{\Pi(I,k,S)}{\mathrm{OPT}(I,k)} \leq \alpha \cdot \frac{\Pi(I',k',S')}{\mathrm{OPT}(I',k')}$.

A *reduction rule* is the execution of the reduction algorithm on an instance. A reduction rule is said to be applicable on an instance if the output instance is different from the input instance. An *α-approximate kernelization (or α-approximate kernel)* for Π is an α-approximate polynomial-time preprocessing algorithm such that the size of the output instance is upper bounded by a computable function $g : \mathbb{N} \times \mathbb{N}$. In classical kernelization, often we apply reduction rules several times to reduce the given instance. This however breaks down in lossy kernelization, since each application of a reduction rule introduces a "gap" between the approximation quality of the kernel and that of the original instance. This is remedied by introducing α-strict kernelization and α-safe reduction rules. An α-approximate kernelization is said to be *strict* if $\frac{\Pi(I,k,s)}{\text{OPT}(I,k)} \leq \max\{\frac{\Pi(I',k',s')}{\text{OPT}(I',k')}, \alpha\}$. A reduction rule is said to be α-*safe* for Π if there is a solution lifting algorithm, such that the rule together with this algorithm constitutes a strict α-approximate polynomial-time preprocessing algorithm for Π. A reduction rule is *safe* if it is 1-safe. A *polynomial-size approximate kernelization scheme (PSAKS)* for Π is a family of α-approximate polynomial kernelization algorithms for each $\alpha > 1$. The size of an output instance of a PSAKS, when run on (I, k) with approximation parameter α, must be upper bounded by $f(\alpha)k^{g(\alpha)}$ for some functions f and g independent of $|I|$ and k. We encourage the reader to see [18] for a more comprehensive discussion of these ideas and definitions.

In [18], the authors exhibit lossy kernels for several problems which do not admit a classical kernelization, such as Connected Vertex Cover, Disjoint Cycle Packing and Disjoint Factors, admit polynomial lossy kernels. They also develop a lower bound framework for lossy kernels, by extending the lower bound framework of classical kernelization. They then show that Longest Path does not admit a lossy kernel of polynomial size unless NP$\subseteq$ coNP/poly. In this paper, we investigate several other problems in the framework the lossy kernelization. In particular, we design lossy polynomial kernels for several *graph contraction* problems which do not admit classical polynomial kernels under well known complexity theory conjectures. These problems are defined as follows. For a graph class $\mathcal{G}$, the $\mathcal{G}$-Contraction problem is to determine if an input graph G can be contracted to some graph $H \in \mathcal{G}$ using at most k edge contractions. These problems are well studied and $\mathcal{G}$-Contraction has been proven to be NP-complete for several classes $\mathcal{G}$ [1, 4, 20, 21]. They have also received a lot of attention in Parameterized Complexity [2, 5, 12, 13, 14, 15, 16, 17, 19]. In this work, we give lossy polynomial kernels for the following problems.

Tree Contraction **Parameter:** k
Input: A graph G and an integer k
Question: Does there exist $F \subseteq E(G)$ of size at most k such that G/F is a tree?

Star Contraction **Parameter:** k
Input: A graph G and an integer k
Question: Does there exist $F \subseteq E(G)$ of size at most k such that G/F is a star?

Out-Tree Contraction **Parameter:** k
Input: A digraph D and an integer k
Question: Does there exist $F \subseteq A(D)$ of size at most k such that D/A is an out-tree?

Cactus Contraction **Parameter:** k
Input: A graph G and an integer k
Question: Does there exist $F \subseteq E(G)$ of size at most k such that G/F is a cactus?

It can be shown that these problems do not admit polynomial kernels, via a parameter preserving reduction from the Red Blue Dominating Set problem. Let us define these terms formally. A *polynomial-time parameter preserving reduction* from a parameterized problem Π_1 to a parameterized problem Π_2 is a polynomial-time function that maps an instance (I_1, k_1) of Π_1 to an instance (I_2, k_2) of Π_2 such that $k_2 = k_1^{\mathcal{O}(1)}$, and (I_1, k_1) is an YES instance of Π_1 if and only if (I_2, k_2) is an YES instance of Π_2. It is known that if Π_1 doesn't admit a polynomial kernel, then neither does Π_2 [3]. Next, let us define the Red Blue Dominating Set problem. The input is a bipartite graph G with bipartition (A, B) and an integer t, this problem asks if B has a subset of at most t vertices that dominates A. This problem is NP-complete [11] and it does not have a polynomial kernel when parameterized by $|A|$ [8]. It was shown that Tree Contraction and Star Contraction do not admit a polynomial kernel, by a polynomial parameter preserving reduction from this problem [16]. We modify the reductions of [16], to show that the remaining two problems do not admit a polynomial kernel as well.

Next, we examine these problems in the framework of lossy kernelization. To this end, we first define a parameterized optimization version of these problems in the following way. Let G be the input (directed) graph G, and F be a subset of its edges.

$$\Pi(G,k,F) = \begin{cases} \infty & \text{if F is not a solution} \\ \min\{|F|, k+1\} & \text{otherwise} \end{cases}$$

We use TC($\cdot$), OTC($\cdot$) and CC($\cdot$) to denote the parameterized optimization version of Tree Contraction, Out-Tree Contraction and Cactus Contraction, respectively. The following theorem is the main result of this paper.

▶ **Theorem 1.1.** *Given a graph (digraph) G on n vertices, an integer k and an approximation parameter $\alpha > 1$, there is an algorithm that runs in $k^{f(\alpha)} n^{\mathcal{O}(1)}$ time and outputs a graph (digraph) G' on $k^{g(\alpha)}$ vertices and an integer k' such that for every $c > 1$, a c-approximate (tree/star/cactus/out-tree contraction) solution for (G', k') can be turned into a $(c\alpha)$-approximate (tree/star/cactus/out-tree contraction) solution for (G, k) in $n^{\mathcal{O}(1)}$. Here $f(\alpha)$ and $g(\alpha)$ are constants depending on α.*

2 Preliminaries

An undirected graph is a pair consisting of a set V of vertices and a set E of edges where $E \subseteq V \times V$. An edge is specified as an unordered pair of vertices. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and edges respectively. Two vertices u, v are said to be *adjacent* if there is an edge uv in the graph. The neighbourhood of a vertex v, denoted by $N_G(v)$, is the set of vertices adjacent to v and its degree $d_G(v)$ is $|N_G(v)|$. The subscript in the notation for neighbourhood and degree is omitted if the graph under consideration is clear. For a set of edges F, $V(F)$ denotes the set of endpoints of edges in F. For a set $S \subseteq V(G)$, $G - S$ denotes the graph obtained by deleting S from G and $G[S]$ denotes the subgraph of G induced on set S. For graph theoretic terms and notation which are not explicitly defined here, we refer the reader to the book by Diestel [7].

Two non-adjacent vertices u and v are called as *false twins* of each other if $N(u) = N(v)$. A *path* $P = (v_1, \ldots, v_l)$ is a sequence of distinct vertices where every consecutive pair of vertices are adjacent. The vertices of P is the set $\{v_1, \ldots, v_l\}$ and is denoted by $V(P)$. The *length* of a path is $|V(P)| - 1$. A *cycle* is a sequence $(v_1, \ldots, v_l, v_1)$ of vertices such that $(v_1, \ldots, v_l)$ is a path and $v_l v_1$ is an edge. A *leaf* is a vertex of degree 1. A graph is called *connected* if there is a path between any pair of its vertices and it is called *disconnected* otherwise. A *cut vertex* of a connected graph G is a vertex v such that $G - \{v\}$ is disconnected. A graph that has no cut vertex is called *2-connected*. A *component* of a disconnected graph is a maximal connected subgraph. A set $S \subseteq V(G)$ is called a *vertex cover* if for every edge uv, either $u \in S$ or $v \in S$. Further, S is called a *connected vertex cover* if $G[S]$ is connected. A set $I \subseteq V(G)$ of pairwise non-adjacent vertices is called as an *independent set*. A set S of vertices is said to *dominate* another set S' of vertices if for every vertex in S', $N(S') \cap S \neq \emptyset$. A *tree* is a connected acyclic graph. A *star* is a tree in which there is a path of length at most 2 between any 2 vertices. A graph is called a *cactus* if every edge is a part of at most one cycle.

The *contraction* operation of an edge $e = uv$ in G results in the deletion of u and v and the addition of a new vertex w adjacent to vertices that were adjacent to either u or v. Any parallel edges added in the process are deleted so that the graph remains simple. The resulting graph is denoted by G/e. Formally, $V(G/e) = V(G) \cup \{w\} \setminus \{u, v\}$ and $E(G/e) = \{xy \mid x, y \in V(G) \setminus \{u, v\}, xy \in E(G)\} \cup \{wx \mid x \in N_G(u) \cup N_G(v)\}$. For a set of edges $F \subseteq E(G)$, G/F denotes the graph obtained from G by sequentially contracting the edges in F. G/F is oblivious to the contraction sequence. A graph G is *contractible* to a graph T, if T can be obtained from G by a sequence of edge contractions. For graphs G and T with $V(T) = \{t_1, \cdots, t_l\}$, G is said to have a T-*witness structure* $\mathcal{W}$ if $\mathcal{W}$ is a partition of $V(G)$ into l sets and there is a bijection $W : V(T) \mapsto \mathcal{W}$ such that the following properties hold.

- For each $t_i \in V(T)$, $G[W(t_i)]$ is connected.
- For a pair $t_i, t_j \in V(T)$, $t_it_j \in E(T)$ if and only if there is an edge between a vertex in $W(t_i)$ and a vertex in $W(t_j)$ in G.

The sets $W(t_1), \cdots, W(t_l)$ in $\mathcal{W}$ are called *witness sets*. Moreover, G is contractible to T if and only if G has a T-witness structure. We associate the T-witness structure $\mathcal{W}$ of G with a set $F \subseteq E(G)$ whose contraction in G results in T by defining F to be the set of the edges of a spanning tree of the $G[W]$ for each $W \in \mathcal{W}$.

▶ **Observation 1.** $|F| = \sum\limits_{W \in \mathcal{W}} (|W| - 1)$.

Then, G is said to be $|F|$-contractible to T and the following observation is easy to verify.

▶ **Observation 2.** For every $W \in \mathcal{W}$, $|W| \leq |F| + 1$. Further, $|\{W \in \mathcal{W} \mid |W| > 1\}| \leq |F|$.

Finally, we observe that t is a leaf in T, then the neighbours of the vertices in $W(t)$ are contained in one witness set.

▶ **Observation 3.** Let t be a leaf in T and t' be its unique neighbour. Then, $\bigcup_{v \in W(t)} N_G(v) \subseteq W(t') \cup W(t)$.

Proof. Consider a leaf t in T. Assume on the contrary that there exists t' and t'' (distinct from t) such that $N(u) \cap W(t') \neq \emptyset$ and $N(v) \cap W(t'') \neq \emptyset$ for some u and v (not necessarily distinct) in $W(t)$. Then, t has degree at least 2 contradicting the fact that it is a leaf. ◀

We denote the set of integers from 1 to n by $[n]$. We also use the bound, $\frac{x+p}{y+q} \leq \max\{\frac{x}{y}, \frac{p}{q}\}$ for any positive real numbers x, y, p, q, to prove that the reduction rules we define are strict α-approximate for some real number α.

3 Tree Contraction

We begin with the Tree Contraction problem, which admits a $4^k n^{\mathcal{O}(1)}$ algorithm (where n is the number of vertices of the input graph) by using a FPT algorithm for Connected Vertex Cover as a subroutine, and further it does not admit a polynomial kernel unless NP $\subseteq$ coNP/*poly* [16]. This lower-bound also holds for Star Contraction. Before we proceed to describe a PSAKS for these problems, we mention the following simplifying assumption known from [16] which states that, the tree witness structure of a graph can be constructed from the tree witness structures of its 2-connected components.

▶ **Lemma 3.1** ([16]). *A connected graph is k-contractible to a tree if and only if each of its 2-connected components is contractible to a tree using at most k edge contractions in total.*

Observe that there can be at most k 2-connected components in the graph, and we can consider each such component separately. The output of our kernelization algorithm will be a disjoint union of the kernels for each 2-connected component. So from now onwards we assume that the input graph is 2-connected. Next, we make some observations on the tree witness structure of a graph.

▶ **Lemma 3.2.** *Let F be a minimal set of edges of a 2-connected graph G such that G/F is a tree T with $V(T) = \{t_1, t_2, \ldots, t_l\}$. Let $\mathcal{W}$ denote the T-witness structure of G. Then, there exists a set F' of at most $|F|$ edges of G such that G/F' is a tree T' and the T'-witness structure $\mathcal{W}'$ of G satisfies the property that $W'(t) \in \mathcal{W}'$ is a singleton set if and only if t is a leaf in T'.*

Proof. First, we show that every vertex $t \in V(T)$ such that $|W(t)| = 1$ is a leaf in T. Suppose there is a non-leaf t in T such that $W(t) = \{u\}$ for some $u \in V(G)$. Then, $T - \{t\}$ has at least two non-empty subtrees, say T_1 and T_2. Consider $U_1 = \bigcup_{t \in V(T_1)} W(t)$ and $U_2 = \bigcup_{t \in V(T_2)} W(t)$. As $\mathcal{W}$ is the T-witness structure of G, it follows that there is no edge between a vertex in U_1 and a vertex in U_2 in $G - \{u\}$. This contradicts the fact that G is 2-connected. Now, consider a leaf t_i in T such that $|W(t_i)| > 1$. Let t_j be the unique neighbour of t_i. As $t_i t_j \in E(T)$, there exists an edge in G between a vertex in $W(t_i)$ and a vertex in $W(t_j)$. Therefore, $G[W(t_i) \cup W(t_j)]$ is connected. We claim that $G[W(t_i) \cup W(t_j)]$ has a spanning tree which has a leaf from $W(t_i)$. Observe that as $|W(t_i)| > 1$, any spanning tree of $G[W(t_i)]$ has at least 2 leaves. If there is a spanning tree of $G[W(t_i)]$ that has a leaf u which is not adjacent to any vertex in $W(t_j)$, then $G[(W(t_i) \cup W(t_j)) \setminus \{u\}]$ is connected too and u is the required vertex. Otherwise, every leaf in every spanning tree of $G[W(t_i)]$ is adjacent to some vertex in $W(t_j)$ and hence $G[(W(t_i) \cup W(t_j)) \setminus \{u\}]$ is connected for each vertex $u \in W(t_i)$. Therefore, as claimed, $G[W(t_i) \cup W(t_j)]$ has a spanning tree which has a leaf v from $W(t_i)$. Consider the partition $\mathcal{W}' = (\mathcal{W} \cup \{W_v, W_{ij}\}) \setminus \{W(t_i), W(t_j)\}$ of G where $W_v = \{v\}$ and $W_{ij} = (W(t_j) \cup W(t_i)) \setminus \{v\}$. Then, as $N(v) \subseteq W(t_i) \cup W(t_j)$ by Observation 3, it follows that $\mathcal{W}'$ is the T'-witness structure of G such that T' is a tree. Further, T' is the tree obtained from T by adding a new vertex t_{ij} adjacent to $N(t_j)$ and a new vertex t_v adjacent to t_{ij} and then deleting t_i, t_j. This leads to a set F' of at most $|F|$ edges of G such that $T' = G/F'$ is a tree. Repeating this procedure ensures that the leaves of the resulting tree corresponds to singleton witness sets. ◀

Subsequently, we assume that all tree witness structures have this property. Lemma 3.2 immediately leads to the following equivalence of Star Contraction and Connected Vertex Cover.

▶ **Lemma 3.3.** *G has a set $F \subseteq E(G)$ such that G/F is a star if and only if G has a connected vertex cover of size $|F| + 1$.*

Proof. Let F be a set of edges of G such that G/F is a star T. By Lemma 3.2, we can assume that every leaf of T corresponds to a singleton witness set. If T has at most 2 vertices, then the claim trivially holds. Otherwise, T has at least 3 vertices. Let t_0 be the vertex that is adjacent to all other vertices of T and let t_i, t_j be two leaves of T. Let $W(t_i) = \{u\}$ and $W(t_j) = \{v\}$ for some $u, v \in V(G)$. Since $t_i t_j \notin E(G/F)$, we have $uv \notin E(G)$. Hence $G - W(t_0)$ is an independent set and $G[W(t_0)]$ is connected. Further, $|W(t_0)| = |F| + 1$. Thus, $W(t_0)$ is the required connected vertex cover of G. Conversely, consider a connected vertex cover X of G. Consider the partition $\mathcal{W} = X \cup \bigcup_{u \in V(G) \setminus X} \{u\}$. Then, every set in this partition induces a connected subgraph. Further, as $G - X$ is an independent set, for any two parts W and W' (excluding X) in this partition, there is no edge in G between a vertex in W and a vertex in W'. Thus, $\mathcal{W}$ is a T-witness structure of G where T is a star. Moreover, $G/F = T$ where F is the set of edges of a spanning tree of $G[X]$ and hence $|F| = |X| - 1$. ◀

As Connected Vertex Cover has a PSAKS [18], we have the following result.

▶ **Theorem 3.4.** Star Contraction *parameterized by the solution size admits a PSAKS.*

Lemma 3.2 also leads to the following relationship between Tree Contraction and Connected Vertex Cover.

▶ **Lemma 3.5.** *If G is k-contractible to a tree, then G has a connected vertex cover of size at most $2k$.*

Proof. As G is k-contractible to a tree, there exists a (minimal) set of edges F such that $|F| \leq k$ and $T = G/F$ is a tree. Let $\mathcal{W}$ be the T-witness structure of G and $\mathcal{W}'$ denote the set of non-singleton sets in $\mathcal{W}$. Let X denote the set of vertices of G which are in a set in $\mathcal{W}'$. By Lemma 3.2, we can assume that every leaf of T corresponds to a singleton witness set. Let L be the set of leaves of T. Then, $I = \{v \in V(G) \mid v \in W(t), t \in L\}$ is an independent set in G. Thus, X is a vertex cover of G. As $|F| \leq k$, we have $|X| \leq 2k$ as every vertex in X has an edge incident on it that is in F. Finally, since the set of non-leaves of a tree induces a subtree, it follows that $G[X]$ is connected. ◀

Now, we move on to describe a PSAKS for Tree Contraction. We define a partition of vertices of G into the following three parts.

$$H = \{u \in V(G) \mid d(u) \geq 2k+1\}$$

$$I = \{v \in V(G) \setminus H \mid N(v) \subseteq H\}$$

$$R = V(G) \setminus (H \cup I)$$

We define the first reduction rule as follows.

▶ **Reduction Rule 3.1.** If there is a vertex $v \in I$ that has at least $2k+1$ false twins, then delete v. That is, the resultant instance is $(G - \{v\}, k)$.

▶ **Lemma 3.6.** *Reduction Rule 3.1 is safe.*

Proof. Consider a solution F' of the reduced instance (G', k'). If $|F'| \geq k'+1$, then the solution lifting algorithm returns $E(G)$, otherwise it returns $F = F'$. We show that this solution lifting algorithm with the reduction rule constitutes a strict 1-approximate polynomial time preprocessing algorithm. If $|F'| \geq k'+1$ then $\mathrm{TC}(G,k,F) \leq k+1 = \mathrm{TC}(G',k',F')$. Otherwise, $|F'| \leq k$ and let T' be the tree G'/F' and $\mathcal{W}'$ denote the T'-witness structure of G'. Then, as v has at least $2k+1$ false twins, one of these twins, say u, is not in $V(F')$. In other words, there is a vertex t in T' such that $W'(t) = \{u\}$. By Lemma 3.2, t is a leaf. Let t' denote the unique neighbour of t in T'. Then, from Observation 3, $N_{G'}(u) \subseteq W'(t')$. Let T be the tree obtained from T' by adding a new vertex t_v as a leaf adjacent to t'. Since $N_{G'}(u) = N_G(u) = N_G(v)$, all the vertices in $N_G(v)$ are in $W'(t')$. Define the partition $\mathcal{W}$ of $V(G)$ obtained from $\mathcal{W}'$ by adding a new set $\{v\}$. Then, G/F is T and $\mathcal{W}$ is the T-witness structure of G. Hence, $\mathrm{TC}(G,k,F) \leq \mathrm{TC}(G',k',F')$.

Next, consider an optimum solution F^* for (G,k). If $|F^*| \geq k+1$ then $\mathrm{OPT}(G,k) = k+1 \geq \mathrm{OPT}(G',k')$. Otherwise, $|F^*| \leq k$ and let $T = G/F^*$. Let $\mathcal{W}^*$ denote the T-witness structure of G. If there is a leaf t in T such that $W^*(t) = \{v\}$, then F^* is also a solution for

(G', k') and the required relation holds. Otherwise, as v has at least $2k+1$ false twins, one of these twins, say u, is not in $V(F^*)$. That is, there is a leaf t in T such that $W^*(t) = \{u\}$. Define the partition $\mathcal{W}'$ of $V(G)$ obtained from $\mathcal{W}^*$ by replacing u by v and v by u. Then, the set F' of edges of G obtained from F by replacing the edge xv with the edge xu for each x is also an optimum solution for (G,k). Further, it is a solution for (G',k'). Therefore, $\mathrm{OPT}(G',k') \leq \mathrm{OPT}(G,k)$. Hence, $\frac{\mathrm{TC}(G,k,F)}{\mathrm{OPT}(G,k)} \leq \frac{\mathrm{TC}(G',k',F')}{\mathrm{OPT}(G',k')}$. ◀

Given $\alpha > 1$, let d be the minimum integer such that $\alpha \geq \frac{d}{d-1}$. That is, $d = \lceil \frac{\alpha}{\alpha-1} \rceil$. The second reduction rule is the following.

▶ **Reduction Rule 3.2.** If there are vertices $v_1, v_2, \ldots, v_{2k+1} \in I$ and $h_1, h_2, \ldots, h_d \in H$ such that $\{h_1, \ldots, h_d\} \subseteq N(v_i)$ for each $i \in [2k+1]$ then contract all edges in $\tilde{E} = \{v_1 h_i \mid i \in [d]\}$ and reduce the parameter by $d-1$. The resulting instance is $(G/\tilde{E}, k-d+1)$.

▶ **Lemma 3.7.** *Reduction Rule 3.2 is α-safe.*

Proof. Consider a solution F' of the reduced instance (G',k'). If $|F'| \geq k'+1$, then the solution lifting algorithm returns $E(G)$, otherwise it returns $F = F' \cup \tilde{E}$. We will show that this solution lifting algorithm with the reduction rule constitutes a strict α-approximate polynomial time preprocessing algorithm. First, we prove that $\mathrm{TC}(G,k,F) \leq \mathrm{TC}(G',k',F') + d$. If $|F'| \geq k'+1$ then $\mathrm{TC}(G',k',F') = k'+1$. In this case, $F = E(G)$ and $\mathrm{TC}(G,k,F) \leq k+1 = k'+d = \mathrm{TC}(G',k',F') + d - 1$. Consider the case when $|F'| \leq k'$ and let $\mathcal{W}' = \{W'(t_1), W'(t_2), \ldots, W'(t_l)\}$ be the G'/F'-witness structure of G. Let w denote the vertex in $V(G') \setminus V(G)$ obtained by contracting $\tilde{E}$. Without loss of generality, assume that $w \in W'(t_1)$. Define $\mathcal{W} = (\mathcal{W}' \cup \{W_1\}) \setminus \{W'(t_1)\}$ where $W_1 = (W'(t_1) \cup \{v_1, h_1, h_2, \ldots, h_d\}) \setminus \{w\}$. Note that $V(G) \setminus \{v_1, h_1, h_2, \ldots, h_d\} = V(G') \setminus \{w\}$ and hence $\mathcal{W}$ is partition of $V(G)$. Further, $G[W_1]$ is connected as $G'[W'(t_1)]$ is connected. A spanning tree of $G'[W'(t_1)]$ along with $\tilde{E}$ is a spanning tree of $G[W_1]$. Also, $|W_1| = |W'(t_1)| + d$ and any vertex which is adjacent to w in G' is adjacent to at least one vertex in $\{v_1, h_1, h_2, \ldots, h_d\}$ in G. Thus, $\mathcal{W}$ is a G/F-witness structure of G where G/F is a tree isomorphic to G'/F'. Therefore, $\mathrm{TC}(G,k,F) \leq \mathrm{TC}(G',k',F') + d$.

We now show that $\mathrm{OPT}(G',k') \leq \mathrm{OPT}(G,k) - (d-1)$. Let F^* be an optimum solution for (G,k) and $\mathcal{W}$ be a G/F^*-witness structure of G. Let T be G/F^*. If $|F^*| \geq k+1$, then $\mathrm{OPT}(G,k) = k+1 = k'+d \geq \mathrm{OPT}(G',k') + d - 1$. Otherwise, $|F^*| \leq k$ and there is at least one vertex, say v_q in $\{v_1, v_2, \ldots, v_{2k+1}\}$ which is not in $V(F^*)$. By Observation 3, $N(v_q)$ and hence $\{h_1, h_2, \ldots, h_d\}$ are in the same witness set, say $W(t_i)$ where $t_i \in V(T)$. If $v_1 \in W(t_i)$ then $F' = F^* \setminus \tilde{E}$ is solution to (G',k') and so $\mathrm{OPT}(G',k') \leq |F'| = |F^*| - d = \mathrm{OPT}(G,k) - d$. Otherwise, $v_1 \notin W(t_i)$ and let $t_j \in V(T)$ be the vertex such that $v_1 \in W(t_j)$. Then, t_i and t_j are adjacent in T. Define another partition $\mathcal{W}' = \mathcal{W} \cup \{W(t_{ij})\} \setminus \{W(t_i), W(t_j)\}$ of $V(G)$ where $W(t_{ij}) = W(t_i) \cup W(t_j)$. Clearly, $G[W(t_{ij})]$ is connected. Thus, $\mathcal{W}'$ is a G/F-witness structure of G where $|F| = |F^*| + 1$ as $|W(t_i)| - 1 + |W(t_j)| - 1 = (|W(t_{ij})| - 1) - 1$. In particular, G/F is the tree obtained from G/F^* by contracting the edge $t_i t_j$. Finally, without loss of generality $\tilde{E} \subseteq F$ and thus $F' = F \setminus \tilde{E}$ is a solution to (G',k'). Therefore, $\mathrm{OPT}(G',k') \leq |F'| = |F^*| + 1 - d = \mathrm{OPT}(G,k) - d + 1$. Combining these bounds, we have $\frac{\mathrm{TC}(G,k,F)}{\mathrm{OPT}(G,k)} \leq \frac{\mathrm{TC}(G',k',F')+d}{\mathrm{OPT}(G',k')+(d-1)} \leq \max\left\{\frac{\mathrm{TC}(G',k',F')}{\mathrm{OPT}(G',k')}, \alpha\right\}$. ◀

This leads to the following bound.

▶ **Lemma 3.8.** *Suppose G is k-contractible to a tree and neither of the Reduction rules 3.1 and 3.2 are applicable on the instance (G,k). Then, $|V(G)|$ is $\mathcal{O}((2k)^{d+1} + k^2)$.*

Proof. We will bound H, I and R separately in order to bound $V(G)$. By Lemma 3.5, G has a connected vertex cover S of size at most $2k$. As H is the set of vertices of degree at least $2k+1$, $H \subseteq S$ and so $|H| \leq 2k$. Every vertex in R has degree at most $2k$. Therefore, as $S \cap R$ is a vertex cover of $G[R]$, $|E(G[R])|$ is $\mathcal{O}(k^2)$. Also, by the definition of I, every vertex in R has a neighbour in R and hence there are no isolated vertices in $G[R]$. Thus, $|R|$ is $\mathcal{O}(k^2)$. Finally, we bound the size of I. For every set $H' \subseteq H$ of cardinality less than d, there are at most $2k+1$ vertices in I which have H' as their neighbourhood. Otherwise, Reduction Rule 3.1 would have been applied. Hence, there are at most $(2k+1) \cdot \binom{2k}{d-1}$ vertices in I which have degree less than d. Further, for a d-size subset H' of H, there are at most $2k+1$ vertices in I which contain H' in their neighbourhood. Otherwise, Reduction Rule 3.2 would have been applied. As a vertex in I of degree at least d is adjacent to all vertices in at least one such subset of H, there are at most $(2k+1)\binom{2k}{d}$ vertices of I of degree at least d. Therefore, $|I|$ is $\mathcal{O}((2k)^{d+1})$. ◀

Now, we have a PSAKS for the problem.

▶ **Theorem 3.9.** Tree Contraction *admits a strict PSAKS with* $\mathcal{O}((2k)^{\lceil \frac{\alpha}{\alpha-1} \rceil + 1} + k^2)$ *vertices.*

Proof. Given $\alpha > 1$, we choose $d = \lceil \frac{\alpha}{\alpha-1} \rceil$ and apply Reduction Rules 3.1 and 3.2 on the instance as long as they are applicable. The reduction rules can be applied in $\mathcal{O}((2k)^d \cdot n^c)$ time where c is a constant independent of α and n is the number of vertices in the input graph. Then, if the reduced graph G has more than $\mathcal{O}((2k)^{d+1} + k^2)$ vertices, then by Lemma 3.8, $\text{OPT}(G,k)$ is $k+1$ and the algorithm outputs $E(G)$ as the solution. Otherwise, G has $\mathcal{O}((2k)^{d+1} + k^2)$ vertices. ◀

4 Out-Tree Contraction

In this section, we describe a PSAKS for an analogue of Tree Contraction in directed graphs. We first require some terminology on directed graphs. A *directed graph (or digraph)* is a pair consisting of a set V of vertices and a set A of directed edges (arcs) where $A \subseteq V \times V$. An arc is specified as an ordered pair of vertices uv and we say that the arc uv is directed from u to v. Let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of a digraph D. For a vertex $v \in V(D)$, $N^-(v)$ denotes the set $\{u \in V(D) \mid uv \in A(D)\}$ of its in-neighbors and $N^+(v)$ denotes the set $\{u \in V(D) \mid vu \in A(D)\}$ of its out-neighbors. The neighbourhood of a vertex v is the set $N(v) = N^+(v) \cup N^-(v)$. The in-degree of a vertex v, denoted by $d^-(v)$, is $|N^-(v)|$. Similarly, its out-degree is $|N^+(v)|$ which is denoted by $d^+(v)$. The (total) degree of v, denoted by $d(v)$, is the sum of its in-degree and out-degree. A sequence $P = (v_1, \cdots, v_l)$ of distinct vertices of D is called a directed path in D if $v_1v_2, \cdots, v_{l-1}v_l \in A(D)$.

For a digraph D, its underlying undirected graph G_D is the undirected graph on the vertex set $V(D)$ with the edge set $\{uv \mid uv \in A(D)\}$. An out-tree T is a digraph where each vertex has in-degree at most 1 such that G_T is a tree. A vertex v of an out-tree is called a *leaf* if $d^-(v) = 1$ and $d^+(v) = 0$. The *root* of an out-tree is the unique vertex that has no in-neighbour. The contraction of an arc $e = uv$ in D results in the digraph, denoted by D/e, on the vertex set $V' = V(D) \setminus \{u,v\} \cup \{x\}$ with $A(D/e) = \{pq \mid pq \in A(D) \text{ and } p,q \in V'\} \cup \{xz \mid vz \in A(D)\} \cup \{zx \mid zu \in A(D)\} \cup \{xz \mid uz \in A(D)\} \cup \{zx \mid zv \in A(D)\}$. The notion of witness structures and witness sets are extended to digraphs as follows. For digraphs D and T with $V(T) = \{t_1, \cdots, t_l\}$, D is said to have a T-witness structure $\mathcal{W}$ if $\mathcal{W}$ is a partition of $V(D)$ into l sets (called witness sets) and there is a bijection $W : V(T) \mapsto \mathcal{W}$ such that the following properties hold.

- For each $t_i \in V(T)$, $G_D[W(t_i)]$ is connected.
- For a pair $t_i, t_j \in V(T)$, $t_i t_j \in A(T)$ if and only if there is an arc from a vertex in $W(t_i)$ to a vertex in $W(t_j)$ in D.

Analogous to undirected graphs, we associate the T-witness structure $\mathcal{W}$ of G with a set $F \subseteq A(D)$ whose contraction in D results in T by defining F to be the set of the arcs corresponding to the edges of a spanning tree of $G_D[W]$ for each $W \in \mathcal{W}$. Now, we show that similar to Tree Contraction, Out-Tree Contraction also does not admit a polynomial kernel. We modify the reduction known for Tree Contraction to show this hardness.

▶ **Lemma 4.1.** Out-Tree Contraction *does not have a polynomial kernel unless* NP$\subseteq$ coNP*/poly.*

Proof. Consider an instance $(G(A, B), t)$ of Red Blue Dominating Set. We construct an instance (D, k) of Out-Tree Contraction as follows. Let H be the underlying undirected graph of D obtained from G by adding a new vertex u to A that is adjacent to every vertex in B. Also, for every vertex $a \in A$, a set S_a of $k+1$ new vertices that are adjacent to u and a are added to B. Let A' be $A \cup \{u\}$ and B' be the set B along with the $|A| \cdot (k+1)$ new vertices. This completes the construction of H. Also, we set $k = |A| + t$. The digraph D is obtained from H by orientating the edges such that u has no in-neighbours. That is, all edges incident on u are directed away from u and the remaining edges are oriented arbitrarily. We claim that G has a set of at most t vertices in B that dominates A if and only if D is k-contractible to an out-tree.

Suppose there exists a set $S \subseteq B$ of size at most t that dominates A. Let X be $A' \cup S$. Then, $H[X]$ is connected as u is adjacent to all vertices in S and S dominates A. Define a partition $\mathcal{W}$ of $V(D)$ that contains X as one part and a singleton set for every vertex in $V(D) \setminus X$. Now, as $V(D) \setminus X$ is an independent set, it follows that $\mathcal{W}$ is a T-witness structure of D where T is the star obtained from D by contracting all arcs corresponding to the edges of a spanning tree of $H[X]$. As X has at most $|A| + t + 1 = k + 1$ vertices, any spanning tree of $H[X]$ has at most k edges. Thus, D is k-contractible to an out-tree whose underlying undirected graph is a star. Conversely, suppose D is k-contractible to an out-tree T. Let $\mathcal{W}$ be the T-witness structure of D. Let a be a vertex in $A' \setminus \{u\}$. First, we show that there exists $t \in V(T)$ such that $u, a \in W(t)$. Assume on the contrary that $u \in W(t)$ and $a \in W(t')$. Then, as $|S_a| = k+1$, there are $k+1$ cycles in H that contain u, a and a neighbour $b \in B$ of a that pairwise intersect in $\{u, a, b\}$. To destroy all such cycles using k contractions it is necessary that a and u are in the same witness set W. Consequently, it follows that A' is contained in W. As B' is an independent set, $\mathcal{W}$ can be transformed into another partition $\mathcal{W}'$ of $V(D)$ that contains W and a singleton set for every vertex in $B' \setminus W$. Thus, D is k-contractible to an out-tree T' with at least as many vertices as T and $\mathcal{W}'$ is the T'-witness structure of D. Suppose W contains a vertex b' in $B' \setminus B$. Then, by construction, b' is adjacent only to one vertex $a \in A$ and u. Let b be a neighbour of a. Then, $N_D(b') \subseteq N_D(b)$ and so $W' = (W \setminus \{b'\}) \cup \{b\}$ is connected in H and $|W'| \leq |W|$. Thus, replacing W by W' in $\mathcal{W}'$ yields a T''-witness structure of D such that T'' is an out-star with at least as many vertices as T'. By repeating this process, we obtain a T''-witness structure $\mathcal{W}''$ of D with T'' being an out-tree and $\mathcal{W}''$ containing only one non-singleton set W'' such that $W'' \cap (B' \setminus B) = \emptyset$. Then, the set $S = \{v \in B \mid v \in W''\}$ is $W'' \setminus A'$ and as A' is an independent set, S (with at most $k - |A| - 1$ vertices) dominates A in G. ◀

Now, we describe a PSAKS for Out-Tree Contraction. We note that the simplifying assumptions in Tree Contraction, such as ignoring cut vertices and requiring that the leaves of the resultant tree correspond to singleton witness sets, do not hold anymore. Our first reduction rule is based on the observation that the digraph obtained from an out-tree by adding a new vertex as an out-neighbour of a leaf is once again an out-tree.

▶ **Reduction Rule 4.1.** If there is a vertex $v \in V(D)$ with $d^-(v) = 1$ and $d^+(v) = 0$ then delete v. The resulting instance is (D', k') where $D' = D - \{v\}$ and $k' = k$.

▶ **Lemma 4.2.** *Reduction Rule 4.1 is safe.*

Proof. Consider a set $F' \subseteq A(D')$ such that $T = D'/F'$ is an out-tree. If $|F'| \geq k' + 1$, then the solution lifting algorithm returns $A(D)$, otherwise it returns $F = F'$. If $|F'| \geq k' + 1$ then $\mathrm{OTC}(D, k, F) \leq k + 1 = \mathrm{OTC}(G', k', F')$. Otherwise, let $V(T) = \{t_1, \cdots, t_l\}$ and $\mathcal{W}$ denote the T-witness structure of D'. Then, there exists a vertex $t_i \in V(T)$ such that the unique neighbour of v in D is in $W(t_i)$. Define the partition of $V(D)$ as $\mathcal{W}' = \mathcal{W} \cup \{v\}$. Now, no vertex in any set $W \in \mathcal{W}'$ with $W \neq W(t_i)$ contains a vertex that is adjacent to v. Thus, $\mathcal{W}'$ is the D/F-witness structure of D where D/F is the out-tree obtained from T by adding a new vertex t_v as an out-neighbour of t_i. Hence, $\mathrm{OTC}(D, k, F) \leq \mathrm{OTC}(D', k', F')$.

Next, consider an optimum solution F^* to (D, k). If $|F^*| \geq k+1$, then $\mathrm{OPT}(D, k) = k+1$ and by definition, $\mathrm{OPT}(D', k') \leq k' + 1 = k + 1 = \mathrm{OPT}(D, k)$. Otherwise, $|F^*| \leq k$. Let $T = G/F^*$ and $\mathcal{W}^*$ denote the T-witness structure of D. Let $t \in V(T)$ such that $v \in W(t)$. If t is a leaf and $W(t)$ is a singleton set, then F^* is also a solution to (D', k') and $\mathrm{OPT}(D', k') \leq \mathrm{OPT}(D, k)$. Otherwise, as v is a vertex of degree 1, the underlying undirected subgraph of $D[W(t) \setminus \{v\}]$ is connected. Let e be the arc in D that is incident on v. The partition $\mathcal{W}'$ of $V(D')$ obtained from $\mathcal{W}$ by deleting v from $W(t)$ is the $D'/(F^* \setminus e)$-witness structure of D' where $D'/(F^* \setminus e)$ is an out-tree. Thus, $F^* \setminus e$ is a solution to (D', k') and therefore, $\mathrm{OPT}(D', k) \leq \mathrm{OPT}(D, k) - 1$ in this case. Hence, $\frac{\mathrm{OTC}(D,k,F)}{\mathrm{OPT}(D,k)} \leq \frac{\mathrm{OTC}(D',k',F')}{\mathrm{OPT}(D',k')}$. ◀

The operation of subdividing an arc uv in D results in the deletion of the arc uv and the addition of a new vertex w as an out-neighbour of u and an in-neighbour of v. The next reduction rule is based on the observation that subdividing an arc of an out-tree results in another out-tree. To exploit this observation, we need the following lemma.

▶ **Lemma 4.3.** *Suppose D has a directed path $P = (v_0, v_1, \ldots, v_l, v_{l+1})$ with $l > k + 1$ and $d^-(v) = d^+(v) = 1$ for each $v \in V(P)$. Then, no minimal out-tree contraction solution F of D with $|F| \leq k$ contains an arc incident on $V(P) \setminus \{v_0, v_{l+1}\}$.*

Proof. Assume on the contrary that F contains at least one such arc. As there are at least $k + 1$ arcs with endpoints in $V(P) \setminus \{v_0, v_{l+1}\}$ and by the property of F, there is one at least one arc $v_{i-1}v_i \in F$ and $v_iv_{i+1} \notin F$. Let $T = D/F$ with $V(T) = \{t_1, \cdots, t_p\}$ and $\mathcal{W}$ denote the T-witness structure of D. Now, let t and t' denote the vertices of T such that $v_{i-1}, v_i \in W(t)$ and $v_{i+1} \in W(t')$. If $t = t'$, then as $G_D[W(t)]$ is connected, $v_{i-1}, v_i, v_{i+1} \in W(t)$ and $v_iv_{i+1} \notin F$, it follows that $W(t)$ contains the vertices of the subpath $(v_{i+1}, \ldots, v_l, v_{l+1})$ and the vertices of the subpath $(v_0, v_1, \ldots, v_{i-1}, v_i)$. Then, $|W(t)| > k + 1$ which leads to a contradiction. Thus, $t \neq t'$. Now, v_i is not a cut vertex in $G_D[W(t)]$ as there is exactly one edge incident on it. This shows that $G_D[W(t) \setminus v_i]$ is connected. Define $\mathcal{W}' = (\mathcal{W} \setminus \{W(t)\}) \cup \{v_i\} \cup \{W(t) \setminus \{v_i\}\}$. Now, $D/(F \setminus \{v_{i-1}v_i\})$ is the graph formed by subdividing the arc tt' in the out-tree T. Thus, $\mathcal{W}'$ is an out-tree witness structure of D leading to the solution $F \setminus \{v_{i-1}v_i\}$ which contradicts the minimality of F. ◀

▶ **Reduction Rule 4.2.** If there is a directed path $P = (v_0, v_1, \ldots, v_l, v_{l+1})$ with $l > k+2$ and $d^-(v) = d^+(v) = 1$ for each $v \in V(P)$, then replace P by the path $P' = (v_0, v_1, \ldots, v_{k+2}, v_{l+1})$.

Specifically, the resulting instance is $(D', k' = k)$ where D' is the digraph obtained from D by deleting $\{v_{k+3}, \ldots, v_l\}$ and adding the arc $v_{k+2}v_{l+1}$.

We note that this rule can be applied in polynomial time by searching for such a path in the subgraph induced on the vertices of degree 2.

▶ **Lemma 4.4.** *Reduction Rule 4.2 is safe.*

Proof. Consider a minimal set $F' \subseteq A(D')$ such that $T' = D'/F'$ is an out-tree. If $|F'| \geq k' + 1$, then the solution lifting algorithm returns $A(D)$, otherwise it returns $F = F'$. If $|F'| \geq k' + 1$ then $\mathrm{OTC}(D, k, F) \leq k + 1 = \mathrm{OTC}(D', k, F')$. Otherwise, let $V(T') = \{t_1, \cdots, t_r\}$ and $\mathcal{W}'$ denote the T'-witness structure of D'. Then, by Lemma 4.3, F' has no arc incident on $V(P') \setminus \{v_0, v_{l+1}\}$. Therefore, every vertex in $V(P') \setminus \{v_0, v_{l+1}\}$ is in a singleton set of $\mathcal{W}'$. Define $\mathcal{W}$ to be the partition of $V(D)$ that contains every set in $\mathcal{W}'$ and a singleton set W_v for each vertex v in $V(D) \setminus V(D')$. Then, $\mathcal{W}$ is a T-witness structure of D where $T = D/F$ obtained from T' by subdividing some of its edges. As T' is an out-tree, T is an out-tree too. Therefore, $\mathrm{OTC}(D, k, F) \leq \mathrm{OTC}(D', k', F')$.

Next, consider a minimal optimum solution F^* to (D, k). If $|F^*| \geq k + 1$ then $\mathrm{OPT}(D, k) = k + 1$ and by definition, $\mathrm{OPT}(D', k') \leq k' + 1 = k + 1 = \mathrm{OPT}(D, k)$. Otherwise, $|F^*| \leq k$ and let $T = D/F^*$. Let $\mathcal{W}$ denote the T-witness structure of D. By Lemma 4.3, F^* has no edge incident on $V(P) \setminus \{v_0, v_{l+1}\}$. Therefore, every vertex in $V(P) \setminus \{v_0, v_{l+1}\}$ is in a singleton set of $\mathcal{W}$. Define $\mathcal{W}'$ to be the partition of $V(D')$ that contains every set in $\mathcal{W}$ that contain a vertex of D'. Then, $\mathcal{W}'$ is a T'-witness structure of D' where $T' = D'/F^*$. Finally, T' is the graph obtained from T by shortening some of its paths. Hence, T' is an out-tree. Thus, $\mathrm{OPT}(D', k') \leq \mathrm{OPT}(D, k)$. Hence, $\frac{\mathrm{OTC}(D,k,F)}{\mathrm{OPT}(D,k)} \leq \frac{\mathrm{OTC}(D',k',F')}{\mathrm{OPT}(D',k')}$. ◀

Before we describe the next reduction rule, we define the following partition of $V(D)$.

$$I = \{v \in V(D) \mid d^+(v) = 0\}$$

$$H = V(D) \setminus I$$

Now, we apply the following reduction rule on I.

▶ **Reduction Rule 4.3.** If there are vertices $v, v_1, v_2, \ldots, v_{2k+1} \in I$ such that $N^-(v) = N^-(v_1) = \cdots = N^-(v_{2k+1})$, then delete v. The resulting instance is (D', k') where $D' = D - \{v\}$ and $k' = k$.

▶ **Lemma 4.5.** *Reduction Rule 4.3 is safe.*

Proof. Consider a set $F' \subseteq A(D')$ such that $T = D'/F'$ is an out-tree. If $|F'| \geq k' + 1$, then the solution lifting algorithm returns $A(D)$, otherwise it returns $F = F'$. If $|F'| \geq k' + 1$ then $\mathrm{OTC}(D, k, F) \leq k + 1 = \mathrm{OTC}(G', k', F')$. Otherwise, let $V(T) = \{t_1, \cdots, t_p\}$ and $\mathcal{W}$ denote the T-witness structure of D'. Then, as $|V(F')| \leq 2k$, there exists a vertex $t_i \in V(T)$ such that $W(t_i) = \{v_j\}$ for some j. Now, as v_j has no out-neighbour in D', t_i is a leaf in T. Let $t_j t_i \in A(T)$. Then, $N(v_j) \subseteq W(t_j)$. Consider the partition of $V(D)$ defined as $\mathcal{W}' = \mathcal{W} \cup \{v\}$. Now, there is a unique $W \in \mathcal{W}$ such that there exists a vertex $u \in W$ that is a neighbour of v. Thus, $\mathcal{W}'$ is a D/F-witness structure of D where D/F is the out-tree obtained from T by adding t_v as an out-neighbour of t_j. Hence, $\mathrm{OTC}(D, k, F) \leq \mathrm{OTC}(D', k', F')$.

Next, consider an optimal solution F^* of (D, k). If $|F^*| \geq k + 1$ then $\mathrm{OPT}(D, k) = k + 1$ and by definition, $\mathrm{OPT}(D', k') \leq k' + 1 = k + 1 = \mathrm{OPT}(D, k)$. Otherwise, $|F^*| \leq k$ and let $T = G/F^*$. Let $\mathcal{W}^*$ denote the T-witness structure of D. Let $t \in V(T)$ such that $v \in W(t)$. If t is a leaf and $W(t)$ is a singleton set, then F^* is also a solution to (D', k') and $\text{OPT}(D', k') \leq \text{OPT}(D, k)$. Otherwise, as there are at least $2k+1$ vertices with the same neighbourhood as v, there exists one such vertex u for which there exists a vertex $t' \in V(T)$ with $W^*(t') = \{u\}$. As u has no out-neighbours, t' is a leaf. Define the partition $\mathcal{W}'$ of $V(G)$ obtained from $\mathcal{W}^*$ by renaming u by v and v by u. This defines a set of arcs F' obtained from F by replacing the arc xv with the arc xu for each x. Then, F' is an optimum solution for (D, k) and it is a solution for (D', k). Therefore, $\text{OPT}(D', k) \leq \text{OPT}(D, k)$. Hence, $\frac{\text{OTC}(D,k,F)}{\text{OPT}(D,k)} \leq \frac{\text{OTC}(D',k',F')}{\text{OPT}(D',k')}$. ◀

Now, we describe the final reduction rule. Given $\alpha > 1$, let d be the minimum integer such that $\alpha \geq \frac{d}{d-1}$.

▶ **Reduction Rule 4.4.** If there are vertices $v_1, v_2, \ldots, v_{2k+1} \in I$ and $h_1, h_2, \ldots, h_d \in H$ such that $\{h_1, \ldots, h_d\} \subseteq N(v_i)$ for each $i \in [2k+1]$, then contract arcs in $\tilde{A} = \{v_1 h_i \mid i \in [d]\}$ and reduce the parameter by $d-1$. That is, the resulting instance is $(D/\tilde{A}, k-(d-1))$.

▶ **Lemma 4.6.** *Reduction Rule 4.4 is α-safe.*

Proof. Let w denote the vertex in $V(D') \setminus V(D)$ obtained by contracting $\tilde{A}$ in D. Consider a solution F' to the reduced instance (D', k'). If $|F'| \geq k'+1$, then the solution lifting algorithm returns $A(D)$, otherwise it returns $F = F' \cup \tilde{A}$. If $|F'| \geq k'+1$ then $\text{OTC}(D', k', F') = k'+1 = k-d$. In this case, $F = A(D)$ and $\text{OTC}(D, k, F) \leq k+1 = k'+d = \text{OTC}(D', k', F') + d - 1$. Consider the case when $|F'| \leq k'$ and let $\mathcal{W}' = \{W'(t_1), W'(t_2), \ldots, W'(t_l)\}$ be the D'/F'-witness structure of D. Without loss of generality, assume that $w \in W'(t_1)$.

Define $\mathcal{W} = (\mathcal{W}' \cup \{W_1\}) \setminus \{W'(t_1)\}$ where $W_1 = (W'(t_1) \cup \{v_1, h_1, h_2, \ldots, h_d\}) \setminus \{w\}$. Note that $V(D) \setminus \{v_1, h_1, h_2, \ldots, h_d\} = V(D') \setminus \{w\}$ and hence $\mathcal{W}$ is partition of $V(D)$. Further, $G_D[W_1]$ is connected as $G'_D[W'(t_1)]$ is connected. A spanning tree of the latter along with edges $\{v_1 h_i | \, \forall i \in [d]\}$ is a spanning tree of the former. Also, $|W_1| = |W'(t_1)| + d$ and any vertex which is adjacent to w in D' is adjacent to at least one vertex in $\{v_1, h_1, h_2, \ldots, h_d\}$ in D. Thus, $\mathcal{W}$ is a D/F-witness structure of D where D/F is an out-tree. Therefore, $\text{OTC}(D, k, F) \leq \text{OTC}(D', k', F') + d$.

Next, let F^* be an optimum solution for (D, k) and $\mathcal{W}$ be a D/F^*-witness structure of D. Let T denote D/F^*. If $|F^*| \geq k+1$, then $\text{OPT}(D, k) = k+1 = k'+d = \text{OPT}(D', k') + d - 1$. Otherwise, $|F^*| \leq k$ and hence there is at least one vertex, say v_q in $\{v_1, v_2, \ldots, v_{2k+1}\}$ which is not in $V(F^*)$. Then, there is a vertex $t_i \in V(D/F^*)$ such that $W(t_i) = \{v_q\}$. Further t_i is a leaf as v_q has no out-neighbours leading to the existence of a witness set, say $W(t_i)$ where $t_i \in V(T)$, that contains all vertices in $N(v_q)$. Hence $\{h_1, h_2, \ldots, h_d\}$ are in $W(t_i)$. Suppose $v_1 \in W(t_i)$. Let $\tilde{A} = \{v_1 h_i | \, \forall \, i \in [d]\}$. Then, $F' = F^* \setminus \tilde{A}$ is solution to (D', k') and so $\text{OPT}(D', k') \leq |F'| \leq |F^*| - d = \text{OPT}(D, k) - d$. Otherwise, $v_1 \notin W(t_i)$ and then there exists a vertex $t_j \in V(T)$ adjacent to t_i such that $v_1 \in W(t_j)$. Define another partition $\mathcal{W}' = \mathcal{W} \cup \{W(t_{ij})\} \setminus \{W(t_i), W(t_j)\}$ of $V(D)$ where $W(t_{ij}) = W(t_i) \cup W(t_j)$. Clearly, $G_D[W(t_{ij})]$ is connected. Thus, $\mathcal{W}'$ is a D/F-witness structure of D where $|F| = |F^*| + 1$ as $|W(t_i)| - 1 + |W(t_j)| - 1 = (|W(t_{ij})| - 1) - 1$. Further F can be assumed to contain $\tilde{A}$ and $F' = F \setminus \tilde{A}$ is solution to (D', k') leading to $\text{OPT}(D', k') \leq |F'| = |F^*| + 1 - d = \text{OPT}(D, k) - d + 1$. Combining these bounds, we have $\frac{\text{OTC}(D,k,F)}{\text{OPT}(D,k)} \leq \frac{\text{OTC}(D',k',F')+d}{\text{OPT}(D',k')+(d-1)} \leq \max\left\{\frac{\text{OTC}(D',k',F')}{\text{OPT}(D',k')}, \alpha\right\}$. ◀

Now, we prove that the reduction rules described lead to a lossy kernel of polynomial size.

▶ **Lemma 4.7.** *Suppose D is k-contractible to an out-tree and none of Reduction Rules 4.1,4.2,4.3 and 4.4 are applicable on the instance (D, k). Then, $|V(D)|$ is $\mathcal{O}((2k)^{d+1} + k^2)$.*

Proof. For an out-tree T, we define the following sets: $V_1(T) = \{t \in V(T) \mid d(t) = 1\}$, $V_2(T) = \{t \in V(T) \mid d(t) = 2\}$ and $V_3(T) = \{t \in V(T) \mid d(t) \geq 3\}$. Suppose D is contractible to the out-tree T with $V(T) = \{t_1, \cdots, t_l\}$. Let $\mathcal{W}$ denote the T-witness structure of D. Let L denote the set of the leaves in T. Let $t \in L$ and $t't \in A(T)$. Then, $\bigcup_{v \in W(t)} N(v) \subseteq W(t) \cup W(t')$. Further, either $|W(t)| > 1$ or $|W(t')| > 1$. Otherwise, Reduction Rule 4.1 would have been applied. Let L_s denote $\{t \in L \mid |W(t)| = 1\}$, the set of leaves of T that correspond to singleton witness sets in $\mathcal{W}$.

Consider the subtree $T' = T - L_s$. Then, $H \subseteq H'$ where $H' = \{v \in V(D) \mid \exists t \in V(T'), v \in W(t)\}$. We now bound the set H by bounding H'. As Reduction Rule 4.1 is not applicable, $|W(t)| > 1$ for every $t \in V(T')$ and thus $|V_1(T')| \leq k$. As the number of vertices of at least 3 in a tree is upper bounded by the number of leaves, we have $|V_3(T')| \leq k$. Let $V_2 = \{t \in V_2(T') \mid |W(t)| = 1\}$. Clearly, $|V_2(T') \setminus V_2| \leq k$. Thus, $|H' \cap \{v \in V(D) \mid v \in W(t), t \in U\}|$ is $\mathcal{O}(k^2)$ where $U = V_1(T') \cup V_3(T') \cup (V_2(T') \setminus V_2)$. We now bound V_2. Every vertex $t \in V_2$ is either the root or an internal vertex of a path between two vertices in $V_1(T') \cup V_3(T') \cup (V_2(T') \setminus V_2)$ whose internal vertices have degree 2 in the digraph. Now, the number of such paths is at most $2k - 1$. Also, as the length of such a path is $\mathcal{O}(k)$, it follows that $|V_2|$ is $\mathcal{O}(k^2)$.

Summarizing these bounds, it follows that $|H'|$ is $\mathcal{O}(k^2)$ and hence $|H|$ is $\mathcal{O}(k^2)$. Next, we bound the size of I. For every set $H'' \subseteq H$ of cardinality less than d, there are at most $2k + 1$ vertices in I which have H'' as their neighbourhood. Otherwise, Reduction Rule 4.3 would have been applicable. Hence, there are at most $(2k + 1) \cdot \binom{2k}{d-1}$ vertices in I which have degree less than d. Every vertex in I of degree at least d is adjacent to all vertices in at least one d-sized subset of H. For such a subset H'' of H, there are at most $2k + 1$ vertices in I which contain H'' in their neighbourhood. Otherwise, Reduction Rule 4.4 would have been applied. Thus, there $\mathcal{O}((2k + 1)\binom{k^2}{d})$ vertices in I of degree at least d. Hence, $|I|$ is upper bounded by $\mathcal{O}(k^{2d+1})$. ◀

Now, we have the following result.

▶ **Theorem 4.8.** Out-Tree Contraction *admits a PSAKS with $\mathcal{O}(k^{2\lceil \frac{\alpha}{\alpha-1} \rceil + 1} + k^2)$ vertices.*

Proof. Given $\alpha > 1$, we choose $d = \lceil \frac{\alpha}{\alpha-1} \rceil$ and apply the Reduction Rules 4.1,4.2,4.3 and 4.4 exhaustively on the instance. All reduction rules can be applied in $\mathcal{O}((2k)^d \cdot n^c)$ time where c is a constant independent of α and n is the number of vertices in the input graph. If the reduced graph D has more than $\mathcal{O}(k^{2d+1} + k^2)$ vertices, then by Lemma 4.7, $\mathrm{OPT}(D, k)$ is at least $k + 1$ and the algorithm outputs $A(D)$ as the solution. Otherwise, D has $\mathcal{O}(k^{2d+1} + k^2)$ vertices. ◀

5 Cactus Contraction

As mentioned earlier, Tree Contraction has been shown not to admit a polynomial kernel unless NP $\subseteq$ coNP/*poly* by a reduction from Red Blue Dominating Set [16]. We modify this reduction to show similar hardness for Cactus Contraction.

▶ **Lemma 5.1.** Cactus Contraction *does not have a polynomial kernel unless* NP$\subseteq$ coNP/*poly.*

Proof. Consider an instance $(G(A,B),t)$ of Red Blue Dominating Set. We construct an instance (H,k) of Cactus Contraction as follows. H is the graph obtained from G by adding a new vertex u to A that is adjacent to every vertex in B. Also, for every vertex $a \in A$, a set S_a of $5k+3$ new vertices that are adjacent to u and a are added to B. Let A' be $A \cup \{u\}$ and B' be the set B along with the $|A| \cdot (5k+3)$ new vertices. This completes the construction of H. Also, we set $k = |A| + t$. We claim that G has a set of at most t vertices in B that dominates A if and only if H is k-contractible to a cactus.

Suppose there exists a set $S \subseteq B$ of size at most t that dominates A. Let X be the set $A \cup S \cup \{u\}$. Then, $H[X]$ is connected as u is adjacent to all vertices in S and S dominates A. Define a partition $\mathcal{W}$ of $V(H)$ that contains X as one part and a singleton set for every vertex in $V(H) \setminus X$. Now, as $V(H) \setminus X$ is an independent set, it follows that $\mathcal{W}$ is a T-witness structure of H where T is the star obtained from H by contracting all edges of a spanning tree of $H[X]$. As X has at most $|A| + t + 1 = k + 1$ vertices, any spanning tree of $H[X]$ has at most k edges. Thus, H is k-contractible to a star (which is a cactus). Conversely, suppose H is k-contractible to a cactus T. Let $\mathcal{W}$ be the T-witness structure of H. Let a be a vertex in $A' \setminus \{u\}$. First, we show that there exists $t \in V(T)$ such that $u, a \in W(t)$. Assume on the contrary that $u \in W(t)$ and $a \in W(t')$. Then, as $|S_a| = (5k+3)$, there exists distinct vertices t_1, t_2, t_3 of T that are different from t and t' such that $S_a \cap t_i \neq \emptyset$ for each $i \in \{1,2,3\}$. Then, $T[\{t, t', t_1, t_2, t_3\}]$ has a pair of cycles that intersect at more than one vertex leading to a contradiction. Therefore, u and a are in the same witness set W. Consequently, it follows that the vertices in A' are in W. Further as B' is an independent set, $\mathcal{W}$ can be transformed into another partition $\mathcal{W}'$ of $V(H)$ that contains W and a singleton set for every vertex in $B' \setminus W$. Now, it follows that H is k-contractible to a star T' and $\mathcal{W}'$ is the T'-witness structure of H. Moreover, T' has at least as many vertices as T. Suppose W contains a vertex b' in $B' \setminus B$. Then, by construction, b' is adjacent only to one vertex $a \in A$ and u. Let $b \in B$ be a neighbour of a. Then, $N_H(b') \subseteq N_H(b)$ and so $W' = (W \setminus \{b'\}) \cup \{b\}$ is connected and $|W'| \leq |W|$. Thus, replacing W by W' in $\mathcal{W}'$ yields a T''-witness structure of H such that T'' is a star with at least as many vertices as T'. By repeating this process, we obtain a T''-witness structure $\mathcal{W}''$ of H with T'' being a star and $\mathcal{W}''$ containing only one non-singleton set W'' such that $W'' \cap (B' \setminus B) = \emptyset$. Then, the set $S = \{v \in B \mid v \in W''\}$ is $W'' \setminus A'$ and since A' is an independent set, S (with at most $k - |A| - 1$ vertices) dominates A in G. ◀

Next, we proceed to describe a PSAKS for Cactus Contraction. We first list the following simplifying assumption.

▶ **Lemma 5.2.** *A connected graph is k-contractible to a cactus if and only if each of its 2-connected components is contractible to a cactus using at most k edge contractions in total.*

Proof. We prove the claim by induction on the number of vertices in the graph. The claim holds for a graph on a single vertex and assume that it holds for graphs with lesser than n vertices. Consider a connected graph G on n vertices. Suppose G is k-contractible to a cactus. Then, there is a set $F \subseteq E(G)$ of size at most k such that $T = G/F$ is a cactus. Let $\mathcal{W}$ be the T-witness structure of G. Let v be a cut vertex in G and let C be a connected component of $G - \{v\}$. Let G_1 denote the subgraph of G induced on $V(C) \cup \{v\}$ and G_2 denote the subgraph of G induced on $V(G) \setminus V(C)$. Then, G_1 and G_2 are connected graphs satisfying $V(G_1) \cap V(G_2) = \{v\}$. Further, the sets $E(G_1)$ and $E(G_2)$ partition $E(G)$. We claim that $G_1/(F \cap E(G_1))$ and $G_2/(F \cap E(G_2))$ are both cactus graphs. Consider the vertex $t_0 \in V(T)$ such that $v \in W(t_0)$. As the deletion of a vertex in $G_2 - \{v\}$ cannot disconnect G_1, every set in $\mathcal{W}_1 = \{W(t) \setminus V(G_2) \mid \ t \neq t_0, W(t) \in \mathcal{W}\} \cup \{W(t_0) \setminus (V(G_2) \setminus \{v\})\}$ induces a connected subgraph of G. Then, $F \cap E(G_1)$ is the associated set of solution edges and $G_1/(F \cap E(G_1))$ is the subgraph of G/F induced on $\{t \in V(T) \mid W(t) \cap V(G_1) \neq \emptyset\}$. Since an induced subgraph of a cactus is also a cactus, it follows that $G_1/(F \cap E(G_1))$ is a cactus. A similar argument holds for $G_2/(F \cap E(G_2))$. As $E(G_1)$ and $E(G_2)$ form a partition of $E(G)$, $|F \cap E(G_1)| + |F \cap E(G_2)| \leq k$. By induction hypothesis, the required claim holds for G_1 and G_2 and the result follows.

Conversely, let $G_1, G_2, \dots G_l$ be the 2-connected components of G and let $F_i \subseteq E(G_i)$ be a set of edges such that G_i/F_i is a cactus and $\sum_{i \in [l]} |F_i| \leq k$. Let $\mathcal{W}_i$ be the G_i/F_i-witness structure of G_i. Define $\mathcal{W} = \bigcup_{i \in [l]} \mathcal{W}_i$. Now, $\mathcal{W}$ is made into a partition of $V(G)$ as follows: if a vertex v is contained in $W(t_1)$ and in $W(t_2)$ then add $W(t_{12}) = W_1 \cup W_2$ to $\mathcal{W}$ and delete both $W(t_1)$ and $W(t_2)$. Then, $F = \bigcup_{i \in [l]} F_i$ contains the edges of a spanning tree of every witness set in $\mathcal{W}$ and $|F| \leq k$. It remains to argue that G/F is a cactus. If G/F is not a cactus, then there exists two cycles C_1, C_2 which share at least two vertices. As any cycle can have vertices from only one 2-connected component of a graph, C_1, C_2 are both in some 2-connected component of G leading to a contradiction. ◀

So, without loss of generality we can assume that the input graph G is 2-connected. Before we proceed to describe the reduction rules, we need to define some additional terminology. The operation of *subdividing* an edge uv results in the graph obtained by deleting uv and adding a new vertex w adjacent to both u and v. The operation of *short-circuiting* a degree 2 vertex v with neighbours u and w results in the graph obtained by deleting v and then adding the edge uw if it is not already present.

▶ **Observation 4.** The following statements hold for a cactus T.

1. Every vertex of degree at least 3 in T is a cut vertex.
2. The graph obtained by subdividing an edge of T is a cactus.
3. The graph obtained by short-circuiting a degree 2 vertex v in T is a cactus.

Next, we make some observations on the cactus witness structure of a graph.

▶ **Lemma 5.3.** *Let F be a minimal set of edges of a 2-connected graph G such that G/F is a cactus T with $V(T) = \{t_1, t_2, \dots, t_l\}$. Let $\mathcal{W}$ be the T-witness structure of G. Then, the following properties hold.*

1. *There exists a set F' of at most $|F|$ edges of G such that G/F' is a cactus and the G/F'-witness structure $\mathcal{W}'$ of G satisfies the property that for every leaf t in G/F', $W'(t) \in \mathcal{W}'$ is a singleton set.*
2. *If $W(t_1) = \{u_1\}, W(t_2) = \{u_2\}, W(t_3) = \{u_3\}$, then there is a vertex $t \in V(T)$ such that $(N(u_1) \cap N(u_2) \cap N(u_3)) \subseteq W(t)$.*
3. *If t is cut vertex in T then $|W(t)| > 1$.*
4. *If $|F| \leq k$ and $d(v) \geq k + 3$, then $|W(t)| > 1$ where $v \in W(t)$ for $t \in V(T)$.*

Proof. Consider a leaf t_i in T such that $|W(t_i)| > 1$. Let t_j be the unique neighbour of t_i. As $t_i t_j \in E(T)$, there exists an edge in G between a vertex in $W(t_i)$ and a vertex in $W(t_j)$. Therefore, $G[W(t_i) \cup W(t_j)]$ is connected. We claim that $G[W(t_i) \cup W(t_j)]$ has a spanning tree which has a leaf from $W(t_i)$. Observe that as $|W(t_i)| > 1$, any spanning tree of $G[W(t_i)]$ has at least 2 leaves. If there is a spanning tree of $G[W(t_i)]$ that has a leaf u which is not adjacent to any vertex in $W(t_j)$, then $G[(W(t_i) \cup W(t_j)) \setminus \{u\}]$ is connected too and u is the required vertex. Otherwise, every leaf in every spanning tree of $G[W(t_i)]$ is adjacent to some vertex in $W(t_j)$ and hence $G[(W(t_i) \cup W(t_j)) \setminus \{u\}]$ is connected for each vertex $u \in W(t_i)$. Therefore, as claimed, $G[W(t_i) \cup W(t_j)]$ has a spanning tree which has a leaf v from $W(t_i)$. Consider the partition $\mathcal{W}' = (\mathcal{W} \cup \{W_v, W_{ij}\}) \setminus \{W(t_i), W(t_j)\}$ of G where $W_v = \{v\}$ and

$W_{ij} = (W(t_j) \cup W(t_i)) \setminus \{v\}$. Then, as $N(v) \subseteq W(t_i) \cup W(t_j)$ by Observation 3, it follows that $\mathcal{W}'$ is the T'-witness structure of G such that T' is a cactus. Further, T' is the cactus obtained from T by adding a new vertex t_{ij} adjacent to $N(t_j)$ and a new vertex t_v adjacent to t_{ij} and then deleting t_i, t_j. This leads to a set F' of at most $|F|$ edges of G such that $T' = G/F'$ is a cactus. Repeating this procedure ensures that the leaves of the resulting cactus corresponds to singleton witness sets. Therefore, the first property holds.

Let X denote $N(u_1) \cap N(u_2) \cap N(u_3)$. If there exists $t \neq t'$ such that $X \cap W(t)$ and $X \cap W(t')$ are non-empty, T contains two cycles (t_1, t, t_2, t', t_1) and (t_1, t, t_3, t', t_1) that share more than one vertex leading to a contradiction. Thus, the second property is satisfied too. The third property holds as if there is a cut vertex t in T such that $W(t) = \{u\}$. Then, $T - \{t\}$ has at least two non-empty connected graphs, say T_1 and T_2. Consider $U_1 = \bigcup_{t \in V(T_1)} W(t)$ and $U_2 = \bigcup_{t \in V(T_2)} W(t)$. As $\mathcal{W}$ is a cactus witness structure of G, it follows that there is no edge between a vertex in U_1 and a vertex in U_2. This implies that u is a cut vertex in G which leads to a contradiction.

Finally, if $\{v\} = W(t)$ for some $t \in V(T)$ and $d(v) \geq k + 3$, then by the earlier claim, t is not a cut vertex in cactus T. Hence, by Observation 4, the degree of t is one or two. The former case leads to the existence of a vertex t' adjacent to t such that $|W(t')| \geq k+3$ and the latter ascertains the existence of vertices t_1, t_2 adjacent to t such that $|W(t_1)| + |W(t_2)| \geq k + 3$. However, as $|F| \leq k$, both the cases leads to a contradiction as $|W(t')| \leq k + 1$ and $|W(t_1)| - 1 + |W(t_2)| - 1 \leq k$. ◀

Subsequently, we assume that all cactus witness structures have these properties.

▶ **Lemma 5.4.** *Suppose G has a path $P = (u_0, u_1, \ldots, u_l, u_{l+1})$ with $l > k + 1$ consisting of vertices of degree 2. Then, no minimal cactus contraction solution F of G with $|F| \leq k$ contains an edge incident on $V(P) \setminus \{u_0, u_{l+1}\}$.*

Proof. Assume on the contrary that F contains at least one such edge. As there are at least $k + 1$ edges with endpoints in $V(P) \setminus \{u_0, u_{l+1}\}$ and by the property of F, there is at least one edge $u_{i-1}u_i \in F$ and $u_iu_{i+1} \notin F$. Let $T = G/F$ with $V(T) = \{t_1, \cdots, t_p\}$ and $\mathcal{W}$ denote the T-witness structure of G. Now, let t and t' denote the vertices of T such that $u_{i-1}, u_i \in W(t)$ and $u_{i+1} \in W(t')$. If $t = t'$, then as $G[W(t)]$ is connected, $u_{i-1}, u_i, u_{i+1} \in W(t)$ and $u_iu_{i+1} \notin F$, it follows that $W(t)$ contains the vertices of the subpath $(u_{i+1}, \ldots, u_l, u_{l+1})$ and the vertices of the subpath $(u_0, u_1, \ldots, u_{i-1}, u_i)$. Then, $|W(t)| > k + 1$ which leads to a contradiction. Thus, $t \neq t'$. Now, u_i is not a cut vertex in $G[W(t)]$ as there is exactly one edge incident on it. This shows that $G[W(t) \setminus \{u_i\}]$ is connected. Define $\mathcal{W}' = (\mathcal{W} \setminus \{W(t)\}) \cup \{u_i\} \cup \{W(t) \setminus \{u_i\}\}$. Then, $\mathcal{W}'$ is a partition of $V(G)$ which is a G/F'-witness structure of G where $F' = F \setminus \{u_{i-1}u_i\}$. Now, G/F' is the graph formed by subdividing the edge tt' in the cactus T and by Observation 4, G/F' is also a cactus. This contradicts the minimality of F.

◀

Now, we are ready to state the first reduction rule.

▶ **Reduction Rule 5.1.** If G has a path $P = (u_0, u_1, \ldots, u_l, u_{l+1})$ such that $l > k + 2$ consisting of vertices of degree 2, then replace P by the path $P' = (u_0, u_1, \ldots, u_{k+2}, u_{l+1})$. In other words, the resulting instance is $(G', k' = k)$ where G' is the graph obtained from G by deleting $\{u_{k+3}, \ldots, u_l\}$ and adding the edge $u_{k+2}u_{l+1}$.

We observe that this rule can be applied in polynomial time by searching for such a path in the subgraph induced on the vertices of degree 2.

▶ **Lemma 5.5.** *Reduction Rule 5.1 is safe.*

Proof. Consider a minimal set $F' \subseteq E(G)$ such that $T' = G'/F'$ is a cactus. If $|F'| \geq k'+1$, then the solution lifting algorithm returns $E(G)$, otherwise it returns $F = F'$. If $|F'| \geq k'+1$ then $\mathrm{CC}(G, k, F) \leq k+1 = \mathrm{CC}(G', k', F')$. Otherwise, let $V(T') = \{t_1, \cdots, t_r\}$ and $\mathcal{W}'$ denote the T'-witness structure of G'. Then, by Lemma 5.4, F' has no edge incident on $V(P) \setminus \{u_0, u_{l+1}\}$. Therefore, every vertex in $V(P') \setminus \{u_0, u_{l+1}\}$ is in a singleton set of $\mathcal{W}'$. Define $\mathcal{W}$ to be the partition of $V(G)$ that contains every set in $\mathcal{W}'$ and a singleton set W_v for each vertex v in $V(G) \setminus V(G')$. Then, $\mathcal{W}$ is a T-witness structure of G where T is G/F that is obtained from T' by subdividing some of its edges. By Observation 4, T is a cactus as T' is a cactus. Therefore, $\mathrm{CC}(G, k, F) \leq \mathrm{CC}(G', k', F')$.

Next, consider a minimal optimum solution F^* to (G, k). If $|F^*| \geq k+1$ then $\mathrm{OPT}(G, k) = k+1$ and by definition, $\mathrm{OPT}(G', k') \leq k'+1 = k+1 = \mathrm{OPT}(G, k)$. Otherwise, $|F^*| \leq k$ and let $T = G/F^*$. Let $\mathcal{W}$ denote the T-witness structure of G. By Lemma 5.4, F^* has no edge incident on $V(P) \setminus \{u_0, u_{l+1}\}$. Therefore, every vertex in $V(P) \setminus \{u_0, u_{l+1}\}$ is in a singleton set of $\mathcal{W}$. Define $\mathcal{W}'$ to be the partition of $V(G')$ that contains every set in $\mathcal{W}$ that contain a vertex of G'. Then, $\mathcal{W}'$ is a T'-witness structure of G' where $T' = G'/F^*$. Finally, T' is the graph obtained from T by short-circuiting some of its edges. Hence, T' is a cactus. Thus, $\mathrm{OPT}(G', k') \leq \mathrm{OPT}(G, k)$. Hence, $\frac{\mathrm{CC}(G,k,F)}{\mathrm{OPT}(G,k)} \leq \frac{\mathrm{CC}(G',k',F')}{\mathrm{OPT}(G',k')}$. ◀

Now, we describe a property of an instance on which the described reduction rule is not applicable.

▶ **Lemma 5.6.** *If G is k-contractible to a cactus and Reduction Rule 5.1 is not applicable on (G, k), then G has a connected vertex cover of size $\mathcal{O}(k^2)$.*

Proof. Suppose G is k-contractible to the cactus T with $V(T) = \{t_1, \cdots, t_l\}$. Let $\mathcal{W}$ denote the T-witness structure of G. Let V_1, V_2, V_3 be the set of vertices of T of degree 1, 2 and at least 3 respectively. By Lemma 5.3, if $t_i \in V_1$ then $|W(t_i)| = 1$. Consider two vertices t_i and t_j in V_1. Let $W(t_i) = \{u\}$ and $W(t_j) = \{v\}$. Then, as $t_it_j \notin E(T)$, we have that $uv \notin E(G)$. As $T[V_2 \cup V_3]$ is connected, it follows that $S = \bigcup_{t \in V_2 \cup V_3} W(t)$ is a connected vertex cover of G. We now argue that $|S|$ is $\mathcal{O}(k^2)$. For every vertex $t \in V_3$, $|W(t)| > 1$ by Observation 4 and Lemma 5.3. Then, there are at most k vertices in V_3 as G is k-contractible. That is, $\bigcup_{t \in V_3} W(t)$ is upper bounded by $2k$. Further, the number of vertices in V_2 that correspond to non-singleton witness sets is also upper bounded by k. Thus, $\bigcup_{|W(t)|>1, t \in V_2} W(t)$ is upper bounded by $2k$. Now, it remains to bound the size of the set $U = \{t \in V_2 \mid \exists u \in V(G), W(t) = \{u\}\}$. Let T' be the graph obtained from T by short-circuiting all vertices in U. Then, by Observation 4, T' is a cactus with at most $2k$ vertices. Observe that the number of paths with vertices from U in T is bounded by $|E(T')|$. Further, the length of each such path is $\mathcal{O}(k)$ as Reduction Rule 5.1 is not applicable. Since the treewidth of a cactus is at most 2, $|E(T')|$ is $|V(T')|$ which is $\mathcal{O}(k)$. Thus, $|U|$ and hence $|S|$ is $\mathcal{O}(k^2)$. ◀

Before, we describe the next reduction rule, we define a partition of $V(G)$ into the following three parts.

$$H = \{u \in V(G) \mid d(u) \geq k+3\}$$

$$I = \{v \in V(G) \setminus H \mid N(v) \subseteq H\}$$

$$R = V(G) \setminus (H \cup I)$$

The next reduction rule is the following.

▶ **Reduction Rule 5.2.** If there is a vertex $v \in I$ that has at least $2k+3$ false twins, then delete v. That is, the resultant instance is $(G - \{v\}, k)$.

▶ **Lemma 5.7.** *Reduction Rule 5.2 is safe.*

Proof. Consider a solution F' of the reduced instance (G', k'). If $|F'| \geq k'+1$, then the solution lifting algorithm returns $E(G)$, otherwise it returns $F = F'$. If $|F'| \geq k'+1$ then $\mathrm{CC}(G,k,F) \leq k+1 = \mathrm{CC}(G',k,F')$. Otherwise, $|F'| \leq k$ and let T' denote the cactus G'/F' and $\mathcal{W}'$ denote the T'-witness structure of G'. Then, as v has at least $2k+3$ false twins, at least three of these twins, say u_1, u_2, u_3, are not in $V(F')$. By Lemma 5.3, there exists $t_i \in V(T')$ such that $N_{G'}(u_1) \subseteq W'(t_i)$. Let T be the cactus obtained from T' by adding a new vertex t_v as a leaf adjacent to t_i. Since $N_{G'}(u_1) = N_G(u_1) = N_G(v)$, all the vertices in $N_G(v)$ are in $W'(t_i)$. Define the partition $\mathcal{W}$ of $V(G)$ obtained from $\mathcal{W}'$ by adding the new witness set $\{v\}$. Then, T is G/F and $\mathcal{W}$ is the T-witness structure of G. Hence, $\mathrm{CC}(G,k,F) \leq \mathrm{CC}(G',k',F')$.

Next, consider an optimum solution F^* for (G,k). If $|F^*| \geq k+1$ then by definition, $\mathrm{OPT}(G',k') \leq k'+1 = k+1 = \mathrm{OPT}(G,k)$. Otherwise, $|F^*| \leq k$ and let T be the cactus G/F^*. Let $\mathcal{W}^*$ denote the T-witness structure of G. By a similar argument as above, we know that there exists $t_j \in V(T)$ such that $N(v) \subseteq W(t_j)$. Let $t \in V(T)$ such that $v \in W(t)$. If $W(t) = \{v\}$ and t is a leaf in T then F^* is also a solution for (G',k') and the required relation holds. Otherwise, as v has at least $2k+3$ false twins, at least one of them, say u, is in a singleton witness set. That is, there exists a vertex t' in T such that $W(t') = \{u\}$. Define the partition $\mathcal{W}'$ of $V(G)$ obtained from $\mathcal{W}^*$ by renaming u by v and v by u. This leads to a set of edges F' obtained from F by replacing the edge xv with the edge xu for each x. Further, F' is also an optimal solution to (G,k) and it is a solution for (G',k'). Therefore, $\mathrm{OPT}(G',k') \leq \mathrm{OPT}(G,k)$. Hence, $\frac{\mathrm{CC}(G,k,F)}{\mathrm{OPT}(G,k)} \leq \frac{\mathrm{CC}(G',k',F')}{\mathrm{OPT}(G',k')}$. ◀

The final reduction rule is the following. Given $\alpha > 1$, let d be $\lceil \frac{\alpha}{\alpha-1} \rceil$.

▶ **Reduction Rule 5.3.** If there are vertices $v_1, v_2, \ldots, v_{2k+3} \in I$ and $h_1, h_2, \ldots, h_d \in H$ such that $\{h_1, \ldots, h_d\} \subseteq N(v_i)$ for all $i \in [2k+3]$ then contract all edges in $\tilde{E} = \{v_1 h_i \mid i \in [d]\}$ and reduce the parameter by $d-1$. The resulting instance is $(G/\tilde{E}, k-d+1)$.

▶ **Lemma 5.8.** *Reduction Rule 5.3 is α-safe.*

Proof. Consider a solution F' of the reduced instance (G',k'). If $|F'| \geq k'+1$, then the solution lifting algorithm returns $E(G)$, otherwise it returns $F = F' \cup \tilde{E}$. If $|F'| \geq k'+1$ then $\mathrm{CC}(G',k',F') = k'+1 = k-d$. In this case, $F = E(G)$ and $\mathrm{CC}(G,k,F) \leq k+1 = k'+d = \mathrm{CC}(G',k',F') + d - 1$. Consider the case when $|F'| \leq k'$ and let $\mathcal{W}' = \{W'(t_1), W'(t_2), \ldots, W'(t_l)\}$ be the G'/F'-witness structure of G. Let w denote the vertex in $V(G') \setminus V(G)$ obtained by contracting $\tilde{E}$. Without loss of generality, assume that $w \in W'(t_1)$. Define $\mathcal{W} = \mathcal{W}' \cup \{W_1\} \setminus \{W'(t_1)\}$ where $W_1 = W'(t_1) \cup \{v_1, h_1, h_2, \ldots, h_d\} \setminus \{w\}$. Note that $V(G) \setminus \{v_1, h_1, h_2, \ldots, h_d\} = V(G') \setminus \{w\}$ and hence $\mathcal{W}$ is partition of $V(G)$. Further, $G[W_1]$ is connected as $G'[W'(t_1)]$ is connected. A spanning tree of $G'[W_1]$ along with $\tilde{E}$ is a spanning tree of $G[W'(t'_1)]$. Also, $|W_1| = |W'(t_1)| + d$ and any vertex which is adjacent to w in G' is adjacent to at least one vertex in $\{v_1, h_1, h_2, \ldots, h_d\}$ in G. Thus, $\mathcal{W}'$ is a G/F-witness structure of G where G/F is a cactus. Therefore, $\mathrm{CC}(G,k,F) \leq \mathrm{CC}(G',k',F') + d$.

Let F^* be an optimum solution for (G,k) and $\mathcal{W}$ be the G/F^*-witness structure of G. Let T be G/F^*. If $|F^*| \geq k+1$, then $\mathrm{OPT}(G,k) = k+1 = k'+d = \mathrm{OPT}(G',k') + d - 1$. Otherwise, $|F^*| \leq k$ and so there are at least 3 vertices, say v_p, v_q, v_r in $\{v_1, v_2, \ldots, v_{2k+3}\}$ which are not in $V(F^*)$. That is, they are in singleton witness sets of $\mathcal{W}$. Then, by

Lemma 5.3, $\{h_1, h_2, \ldots, h_d\}$ are in the same witness set, say $W(t_i)$ where $t_i \in V(T)$. Suppose $v_1 \in W(t_i)$ and let $\tilde{E} = \{v_1 h_i \mid i \in [d]\}$. Then, $F' = F^* \setminus \tilde{E}$ is solution to (G', k') and so $\text{OPT}(G', k') \leq |F'| \leq |F^*| - d = \text{OPT}(G, k) - d$. Otherwise, $v_1 \notin W(t_i)$ and let $t_j \in V(T)$ be the vertex such that $v_1 \in W(t_j)$. Then, t_i and t_j are adjacent in T. Define another partition $\mathcal{W}' = \mathcal{W} \cup \{W(t_{ij})\} \setminus \{W(t_i), W(t_j)\}$ of $V(G)$ where $W(t_{ij}) = W(t_i) \cup W(t_j)$. Clearly, $G[W(t_{ij})]$ is connected. Thus, $\mathcal{W}'$ is a G/F-witness structure of G where $|F| = |F^*| + 1$ as $|W(t_i)| - 1 + |W(t_j)| - 1 = (|W(t_{ij})| - 1) - 1$. Further F can be assumed to contain $\tilde{E}$ and $F' = F \setminus \tilde{E}$ is solution to (G', k') leading to $\text{OPT}(G', k') \leq |F'| = |F^*| + 1 - d = \text{OPT}(G, k) - d + 1$. Combining these bounds, we have, $\frac{\text{CC}(G,k,F)}{\text{OPT}(G,k)} \leq \frac{\text{CC}(G',k',F') + d}{\text{OPT}(G',k') + (d-1)} \leq \max\left\{\frac{\text{CC}(G',k',F')}{\text{OPT}(G',k')}, \alpha\right\}$. ◀

This leads to the following result.

▶ **Lemma 5.9.** *Suppose graph G is k-contractible to a cactus and none of the Reduction Rules 5.1, 5.2 and 5.3 are applicable on the instance (G, k). Then, $|V(G)|$ is $\mathcal{O}((2k)^{d+1} + k^3)$.*

Proof. The set H consists of only vertices of degree at least $k + 3$ and by Lemma 5.3, every vertex in H is incident on some solution edge hence $|H| \leq 2k$. Since Reduction Rule 5.1 is not applicable, by Lemma 5.6, it follows that G has connected vertex cover S of size $\mathcal{O}(k^2)$. Every vertex in R has degree at most $k + 2$. Therefore, if $S \cap R$ is a vertex cover of $G[R]$, then $|E(G[R])|$ is $\mathcal{O}(k^3)$. Also, by the definition of I, every vertex in R has a neighbour in R and hence there are no isolated vertices in $G[R]$. Thus, $|R|$ is $\mathcal{O}(k^3)$. Now, we bound the size of I. For every set $H' \subseteq H$ of cardinality less than d, there are at most $2k + 3$ vertices in I which have H' as their neighbourhood. Otherwise, Reduction Rule 5.2 would have been applicable. Hence, there are at most $(2k + 3) \cdot \binom{2k}{d-1}$ vertices in I which have degree less than d. Every vertex in I of degree at least d is adjacent to all vertices in at least one subset of size d of H. For a such a subset H' of H, there are at most $2k + 3$ vertices in I which have H' in their neighbourhood. Otherwise, Reduction Rule 3.2 would have been applied. Thus, there are at most $(2k + 3)\binom{2k}{d}$ vertices of I of degree at least d. Hence, $|I|$ is $\mathcal{O}((2k)^{d+1})$. ◀

▶ **Theorem 5.10.** Cactus Contraction *admits a strict PSAKS with $\mathcal{O}((2k)^{\lceil \frac{\alpha}{\alpha-1} \rceil + 1} + k^3)$ vertices.*

Proof. Given $\alpha > 1$, we choose $d = \lceil \frac{\alpha}{\alpha-1} \rceil$ and apply Reduction Rules 5.1, 5.2 and 5.3 on the instance as long as they are applicable. The reduction rules can be applied in $\mathcal{O}((2k)^d \cdot n^c)$ time where c is a constant independent of α and n is the number of vertices in the input graph. If the reduced graph G has more than $\mathcal{O}((2k)^{d+1} + k^3)$ vertices, then by Lemma 5.9, $\text{OPT}(G, k)$ is $k + 1$ and the algorithm outputs $E(G)$ as a solution. Otherwise, G has $\mathcal{O}((2k)^{d+1} + k^3)$ vertices. ◀

6 Concluding Remarks

In this work we gave lossy kernels for several graph contraction problems. The running time of our algorithms have a exponential dependence on the the approximation parameter α. A natural question is if this dependence can be improved to a polynomial, or an even better function. It is also an interesting open problem to construct lossy kernel for other problems, even those that admit a classical kernelization for the reasons we discusses in the introduction. It is also an interesting explore if the techniques described in this paper can be extended to give lossy kernels for other graph contraction problems, e.g. contraction to a graph of bounded treewidth, or to an outerplaner graph.

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