Lossy Kernelization for Some Graph Contraction Problems

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Graph Contraction Problems

Graph Contraction Problems

$\mathcal{F}$ is a graph class and G/F is graph obtained from G by contracting edges in F

$\mathcal{F}$-CONTRACTION	**Parameter:** k

Input: A graph G and an integer k

Question: Does there exist $F \subseteq E(G)$ of size at most k such that G/F is in $\mathcal{F}$?

$\mathcal{F}$-Contraction: Parameterized Complexity

[HvtHL+12]	TREE CONTRACTION PATH CONTRACTION	4^k $2^{k+o(k)}$
[GvtHP13]	PLANAR CONTRACTION	FPT
[CG13]	CLIQUE CONTRACTION	$2^{\mathcal{O}(k \log k)}$
[HvtHLP13] [GM13]	BIPARTITE CONTRACTION	FPT $2^{\mathcal{O}(k^2)}$
[LMS13] [CG13]	$P_{\ell+1}$-FREE CONTRACTION C_ℓ-FREE CONTRACTION	$W[2]$-hard $W[2]$-hard
[ALSZ17]	SPLIT CONTRACTION	$W[2]$-hard

$\mathcal{F}$-Contraction: Kernelization

[HvtHL⁺12]	TREE CONTRACTION	No poly-kernel
	PATH CONTRACTION	$\mathcal{O}(k)$

Tree Contraction

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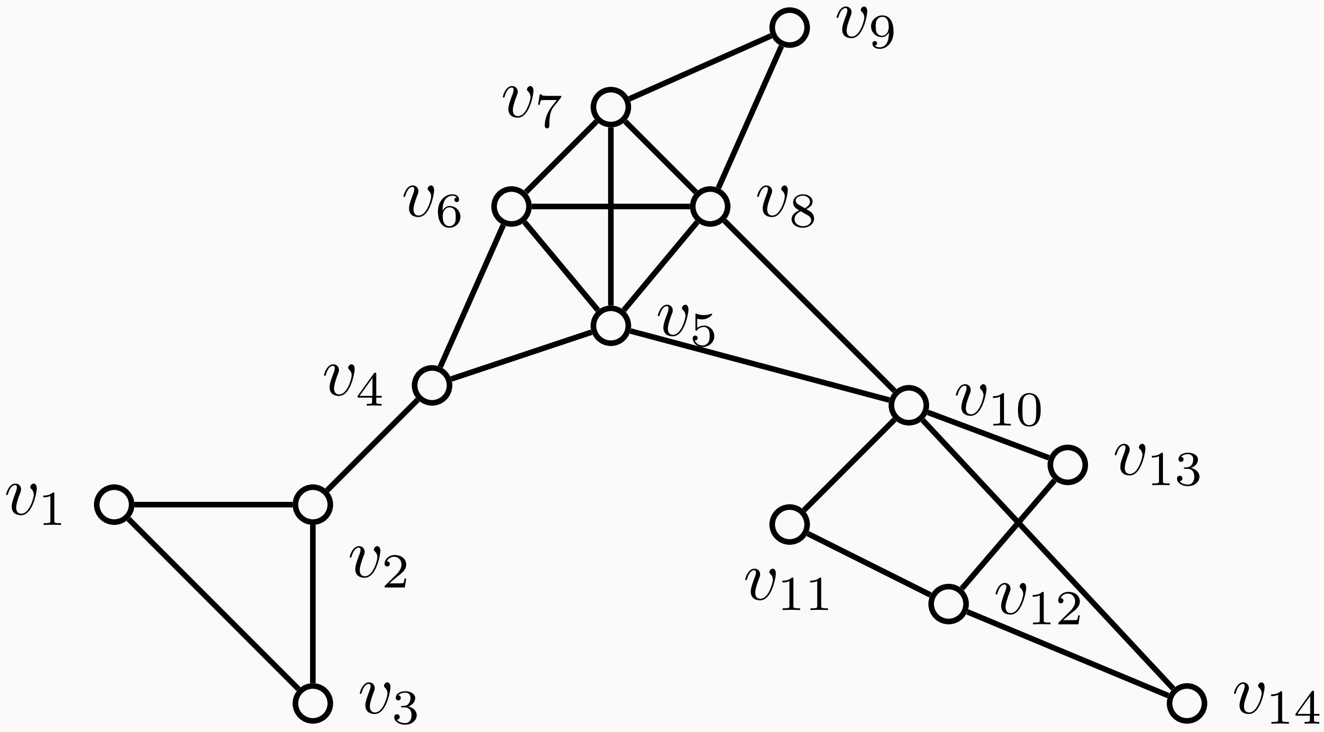

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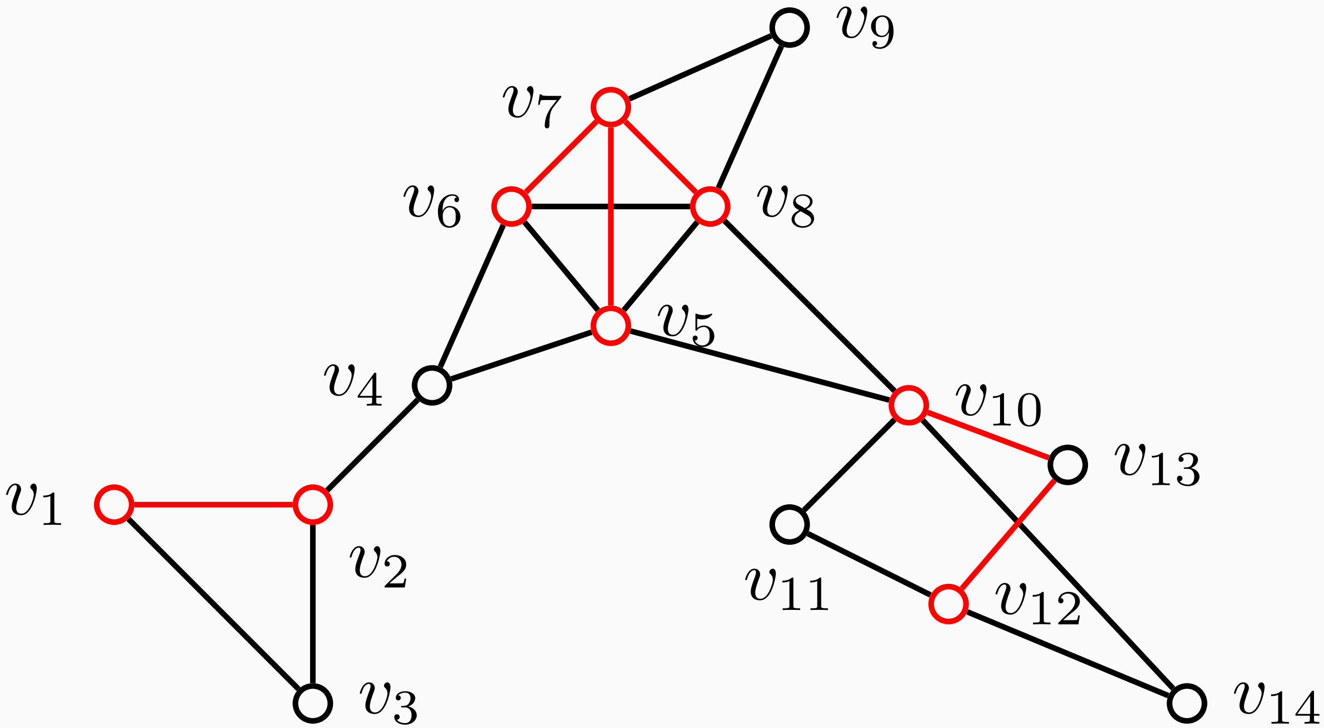

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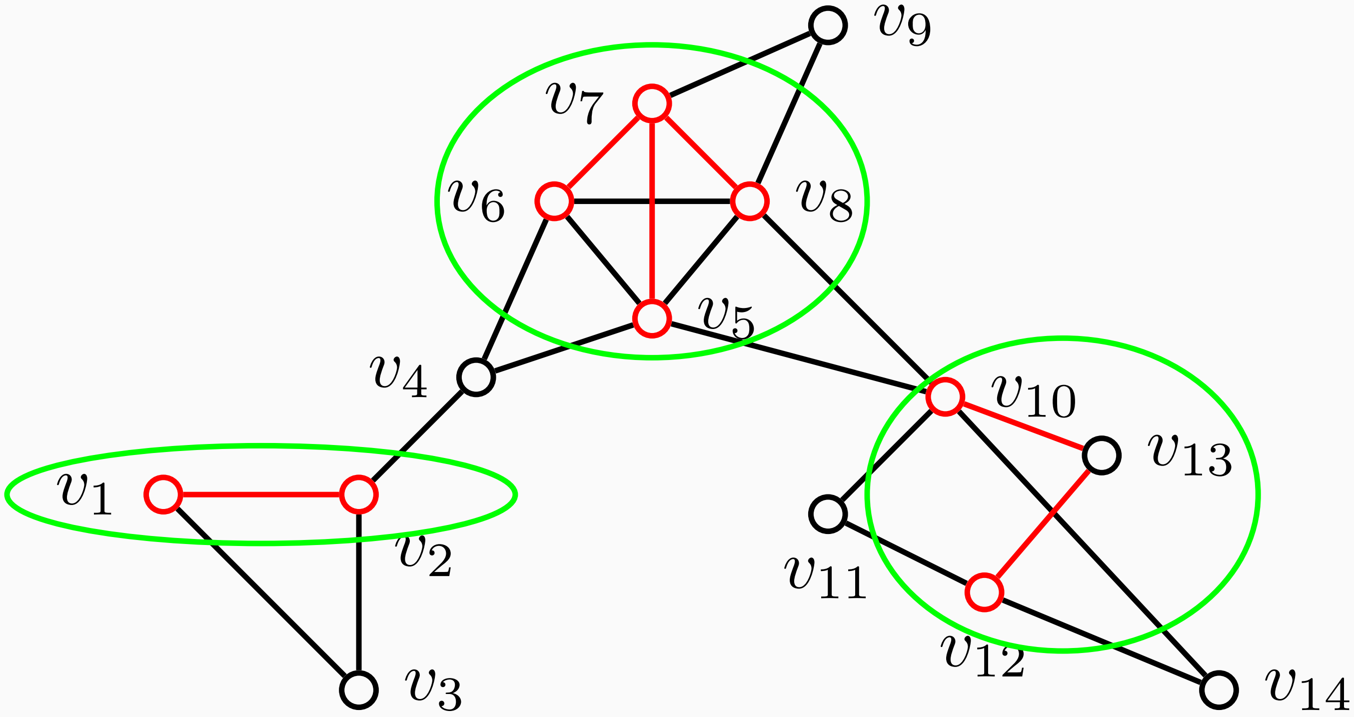

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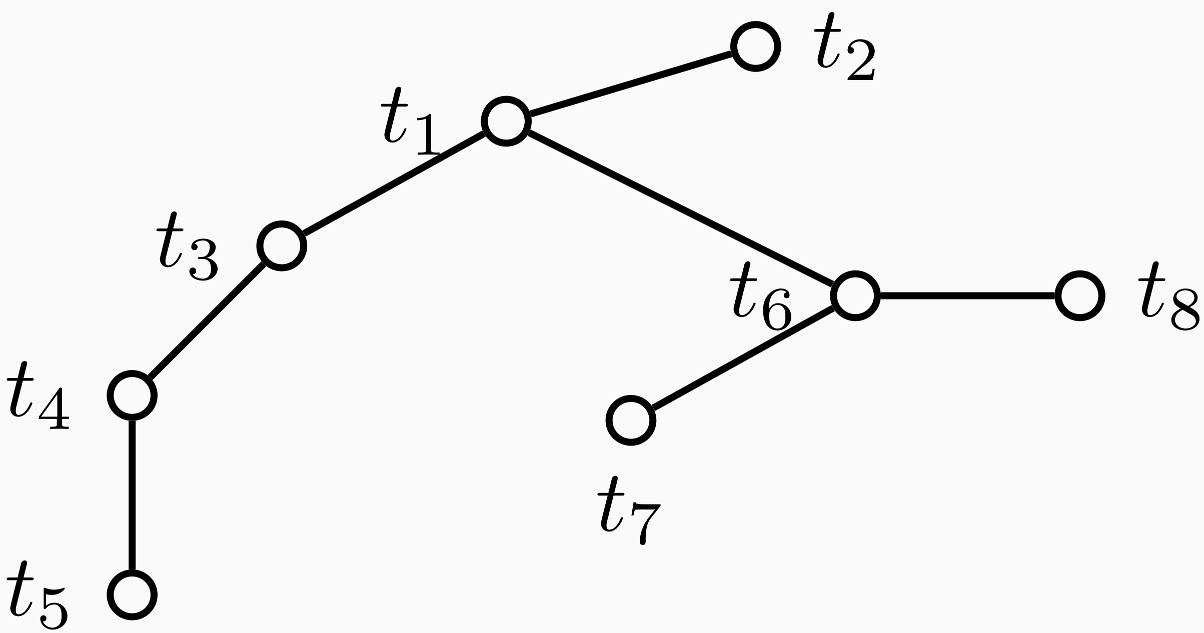

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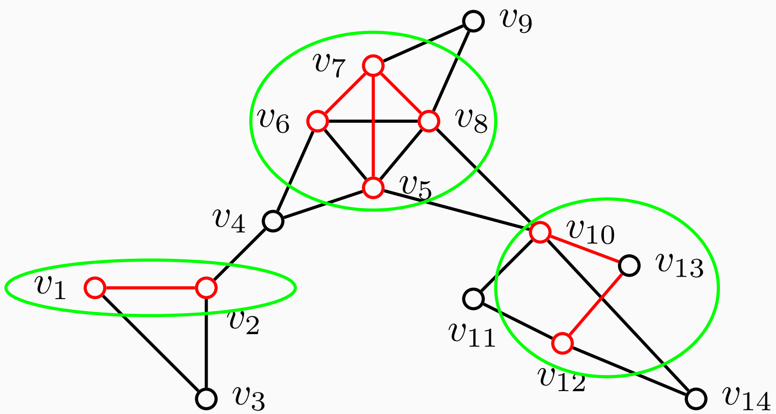

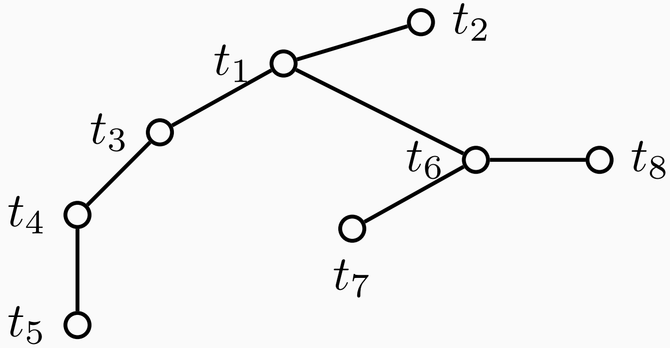

Contraction as a Partition Problem

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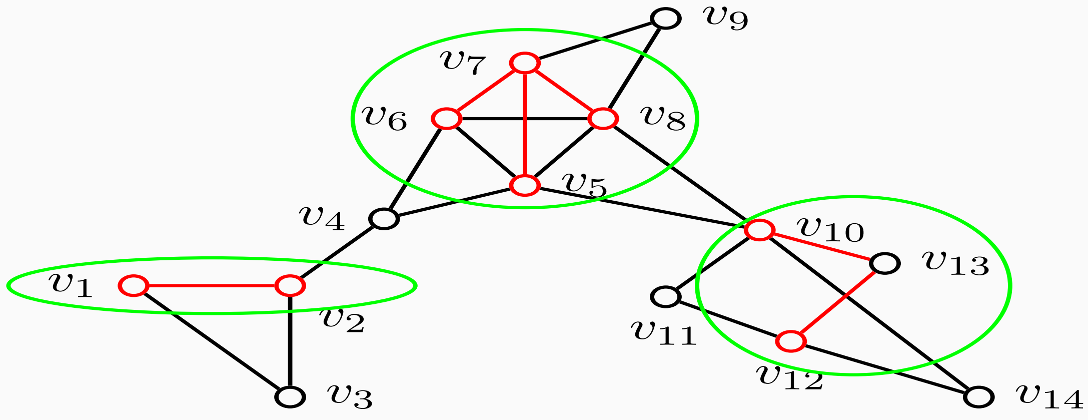

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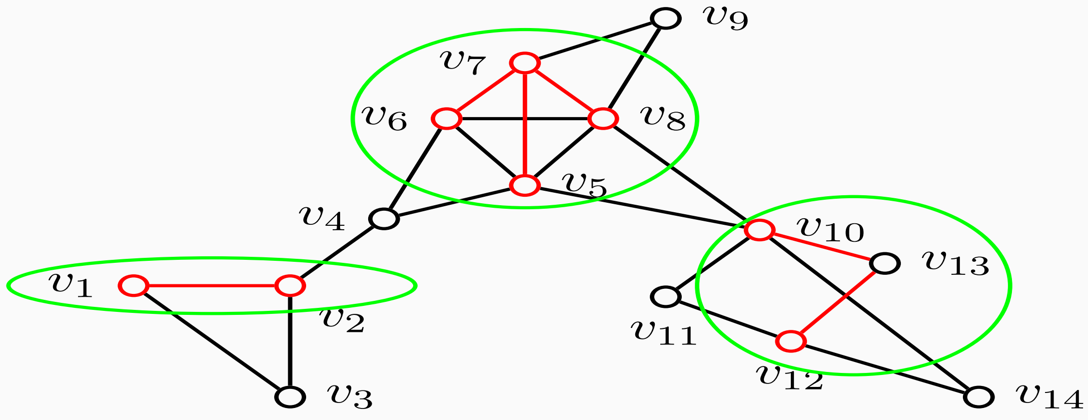

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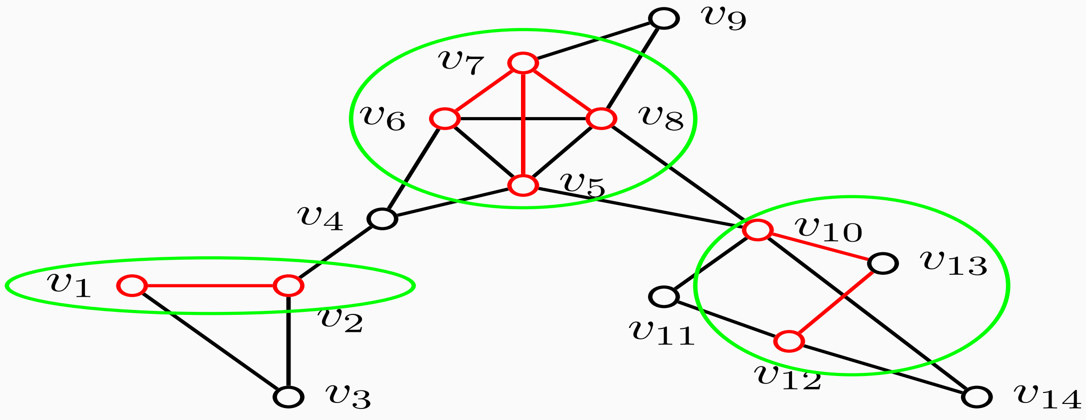
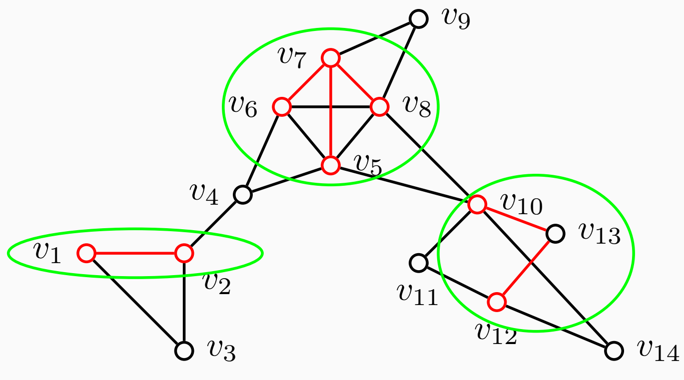
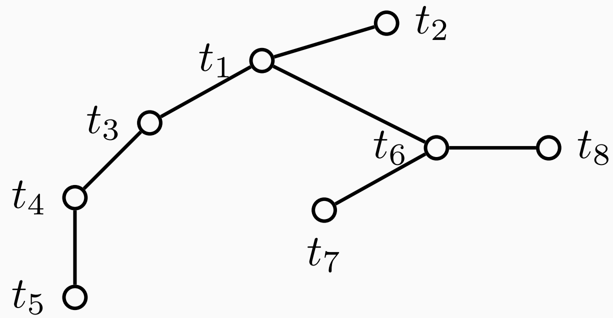


G is contractible to T if there exists a partition of $V(G)$ into $W(t_1), W(t_2), \ldots W(t_{|V(T)|})$ s.t.

- $\forall\ t \in V(T)$, $G[W(t)]$ is connected
- $t_i t_j \in E(T)$ iff $W(t_i)$ and $W(t_j)$ are adjacent in G

Witness Structure : Definition





- $\mathcal{W} = \{W(t) \mid t \in V(T)\}$ is called the T-*witness structure* of G

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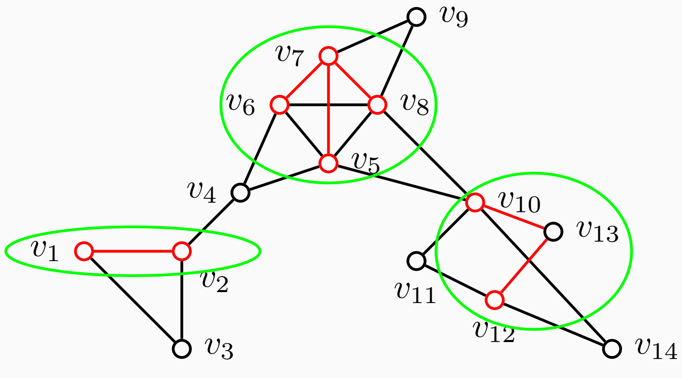

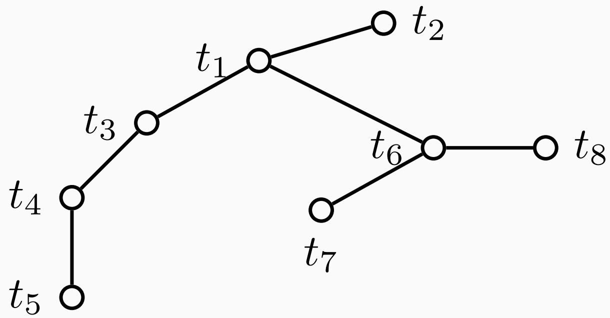

- $\mathcal{W} = \{W(t) \mid t \in V(T)\}$ is called the *T-witness structure* of G
- *Big-witness set* if $|W(t)| > 1$ e.g. $W(t_1), W(t_6), W(t_4)$

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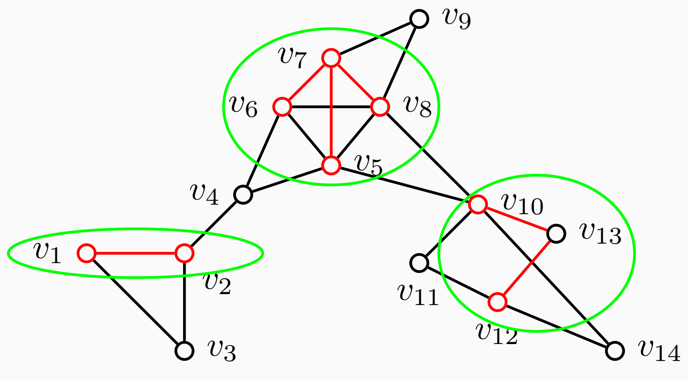

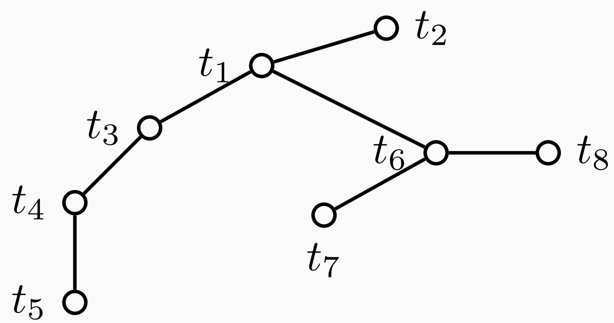

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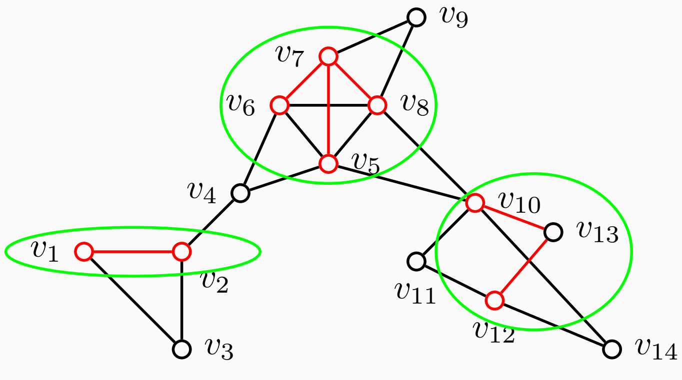
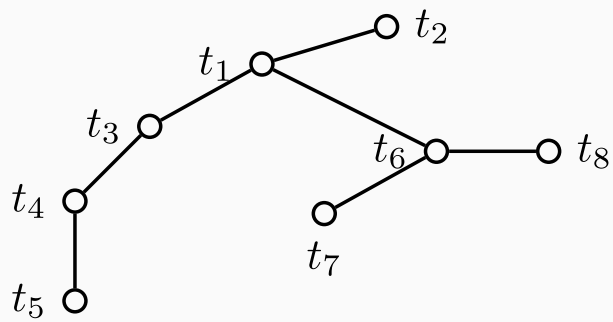
- *Big-witness set* if $|W(t)| > 1$ e.g. $W(t_1), W(t_6), W(t_4)$
- $k = \sum_{t \in V(T)}(|W(t)| - 1)$
  We say G is k-contractible to graph T

Witness Structure : Observations





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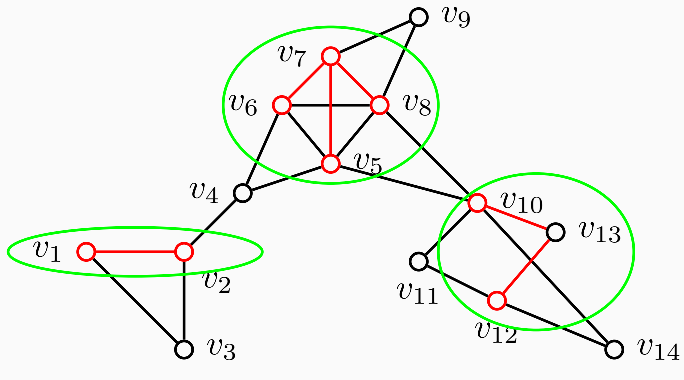

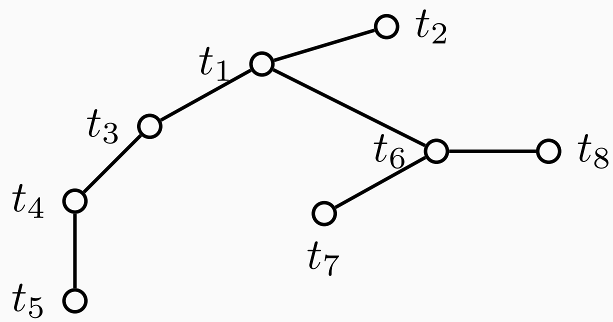

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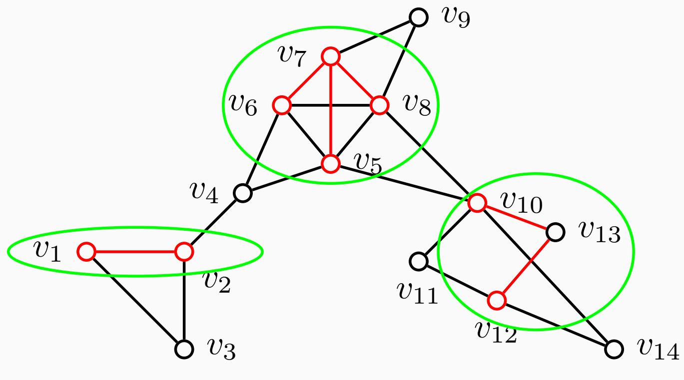

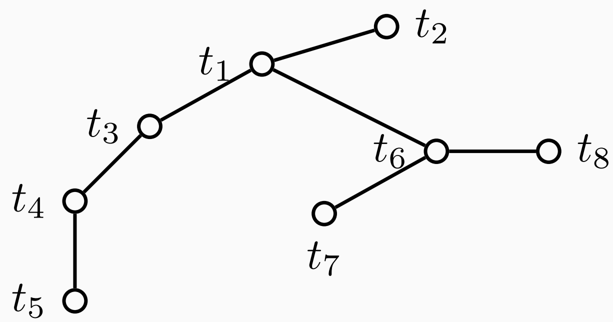

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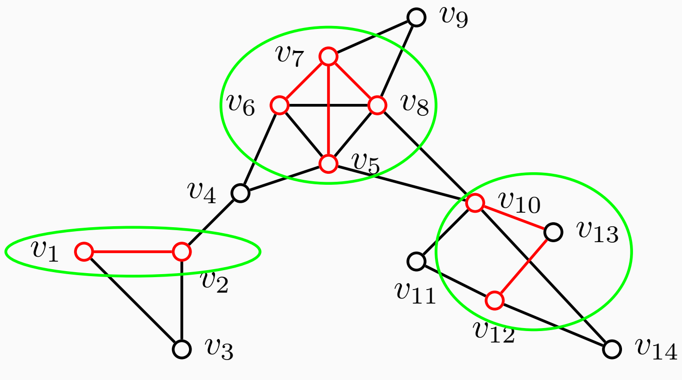

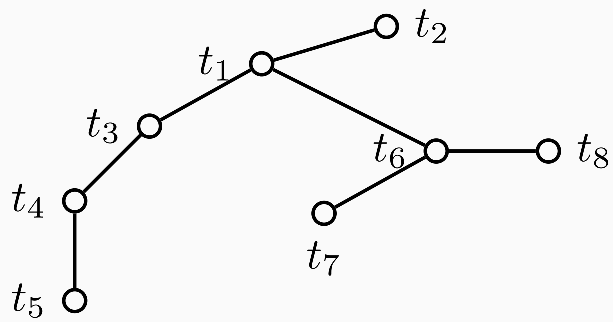

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- No witness set in $\mathcal{W}$ contains more than $k + 1$ vertices;
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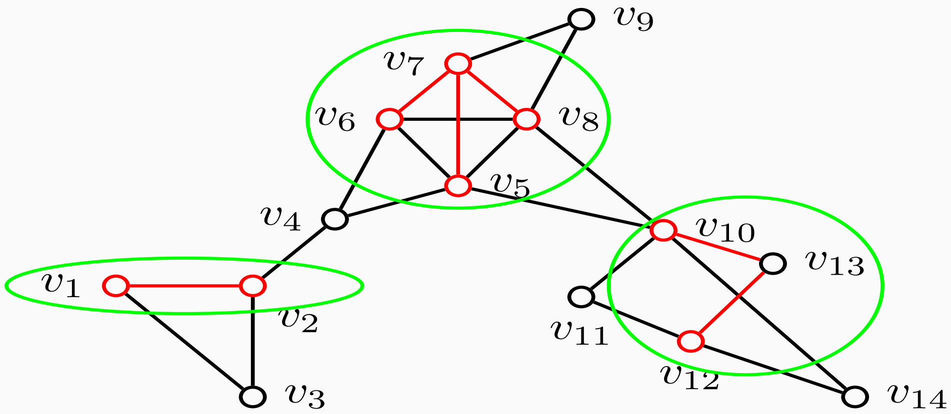
- Union of big witness sets in $\mathcal{W}$ contains at most $2k$ vertices.

Contraction as a Partition Problem



Contraction Problem

- Identify a partition
- Provide connectivity

Lossy Kernelization (Informal Intro)

Kernelization (Reduction Rule)

An algorithm $\mathcal{A}_{Red}$ running in poly(n)

$$(I, k) \longrightarrow (I', k')$$

(I, k) is a yes instance iff (I', k') is a yes instance

Kernelization

In time *poly(n)*, we can find vertices $h_1, h_2, \dots, h_d$ s.t.

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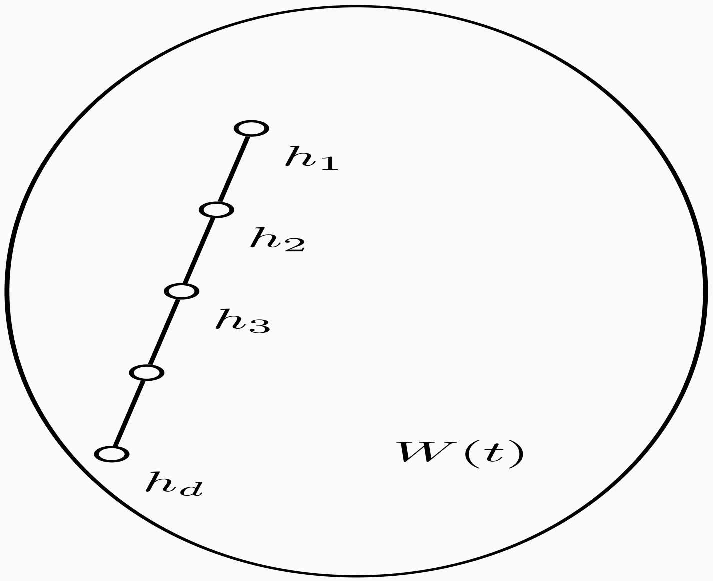

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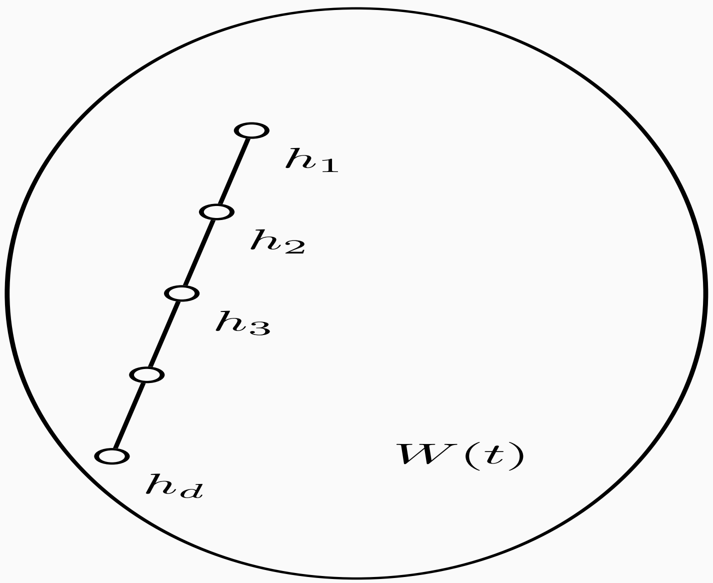

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$\mathcal{A}_{Sol-Lift}$ have access to $\mathcal{A}_{Red}$

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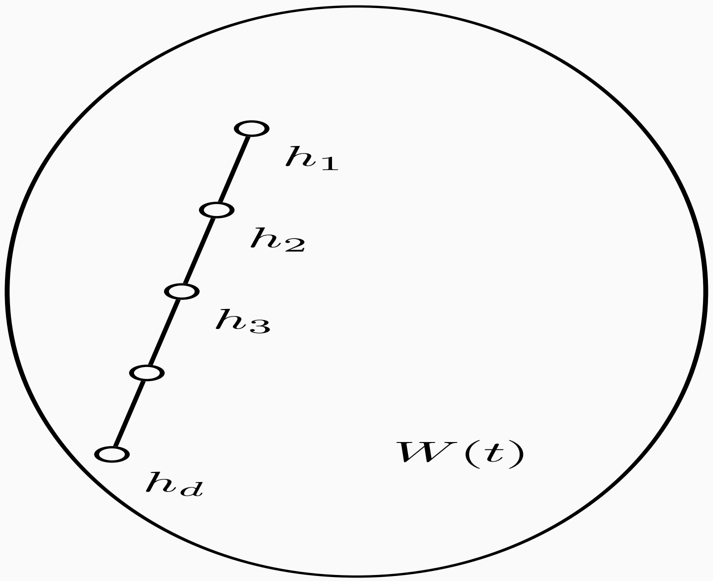

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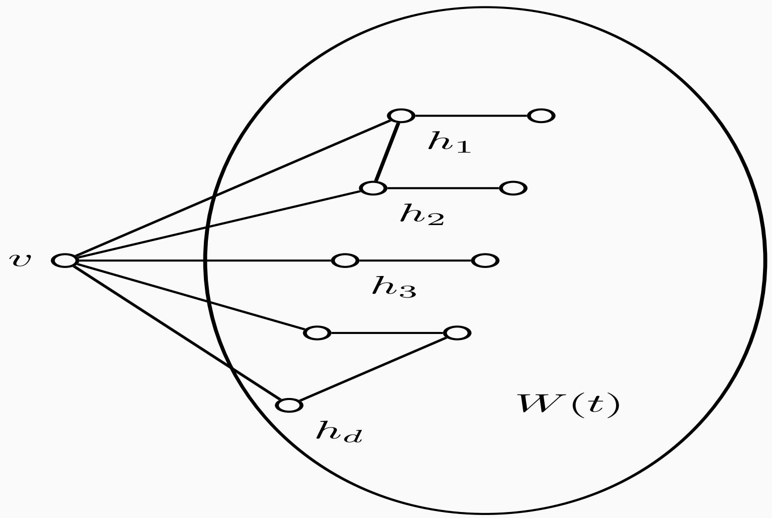

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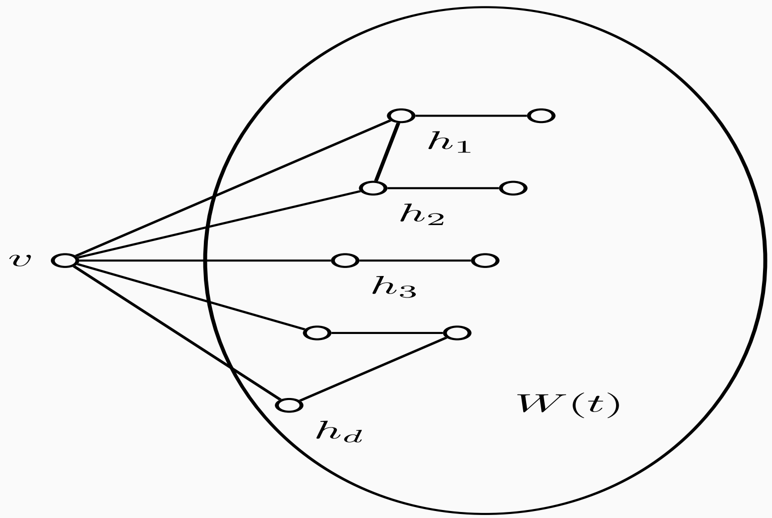

Can we utilize this information to simplify graph?

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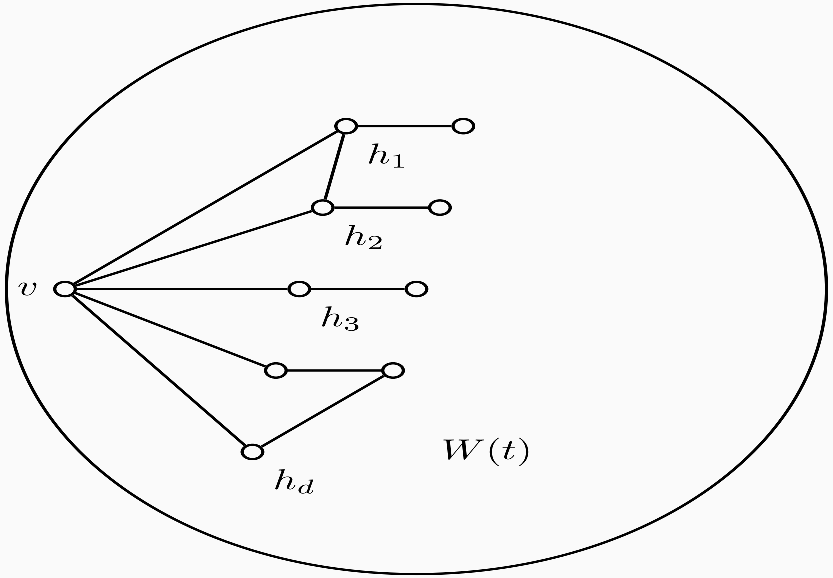

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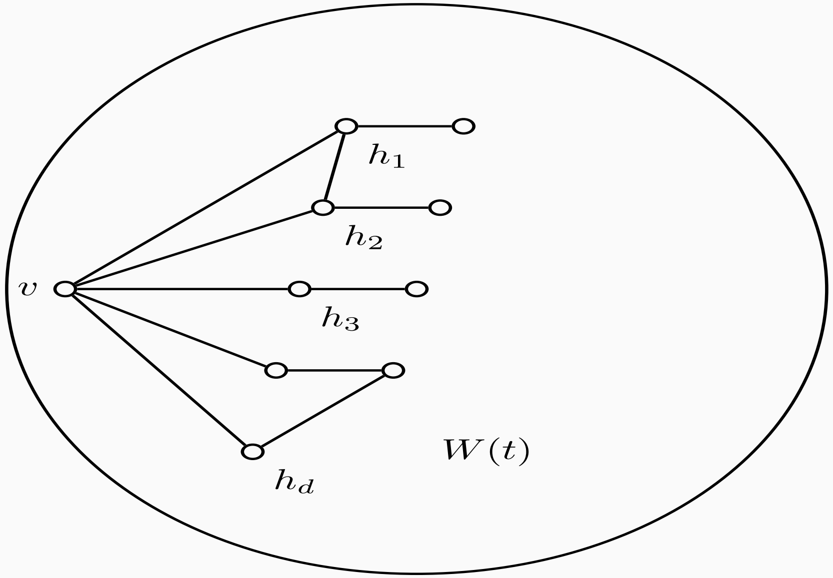

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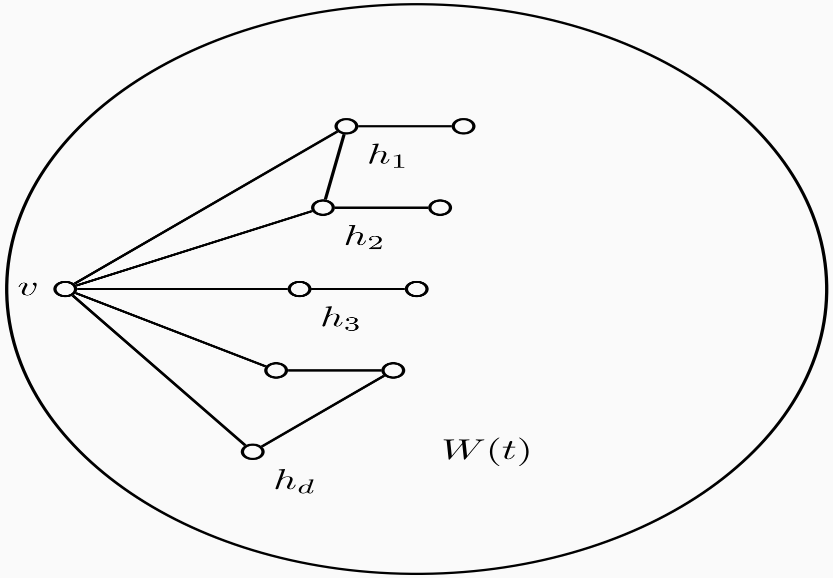

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We contracted d edges but reduced the budget by $d - 1$.

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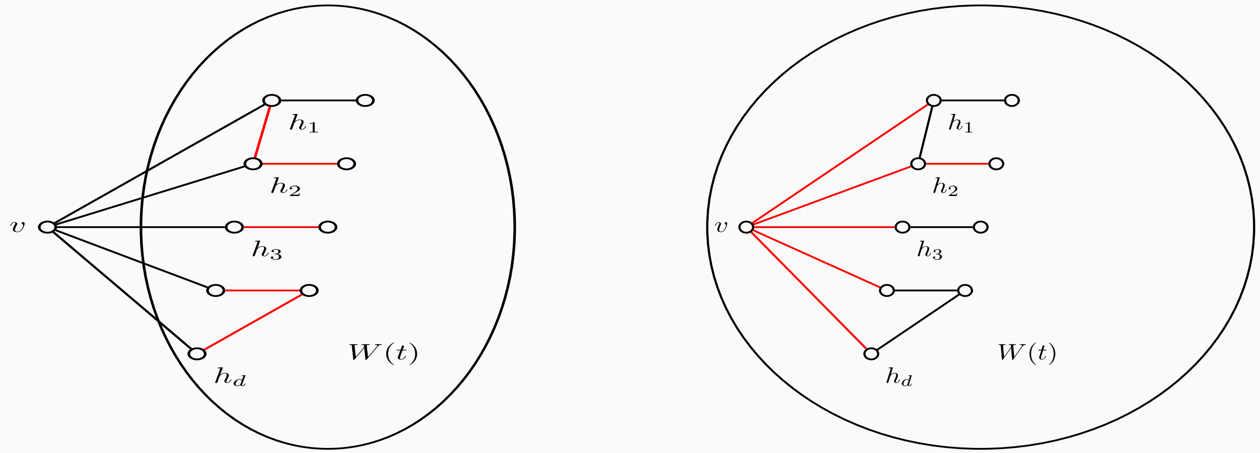

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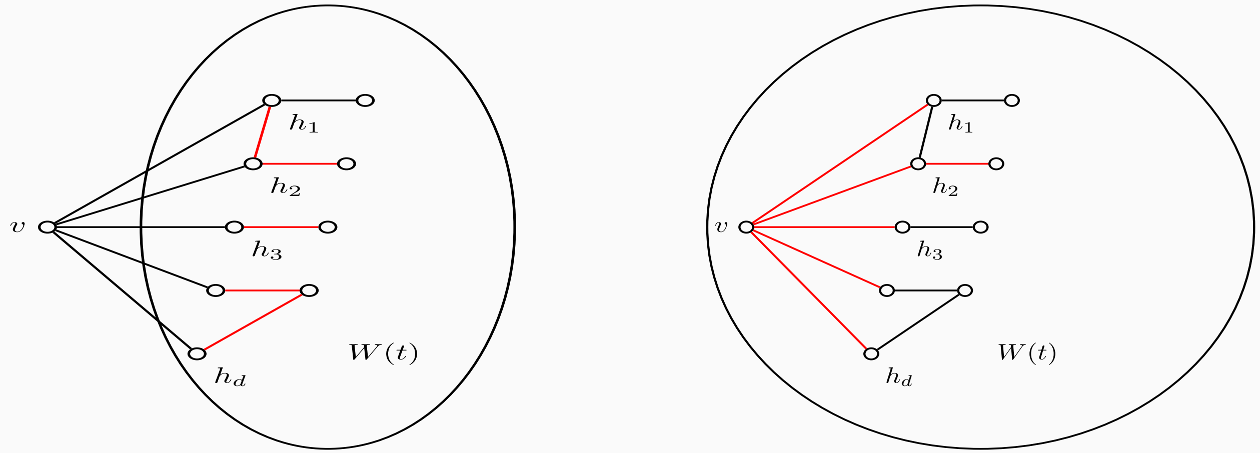

Contracting d-many edges for every $(d-1)$ edges in the solution.

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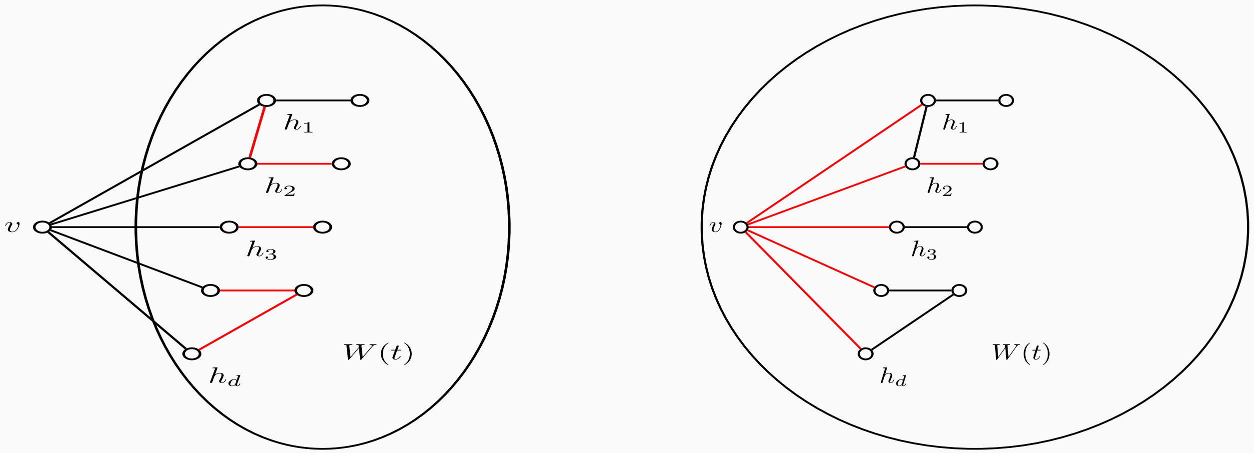

Contracting d-many edges for every $(d-1)$ edges in the solution. The number of edges contracted in this process is $\frac{d}{d-1} = \alpha$ times that of optimum solution

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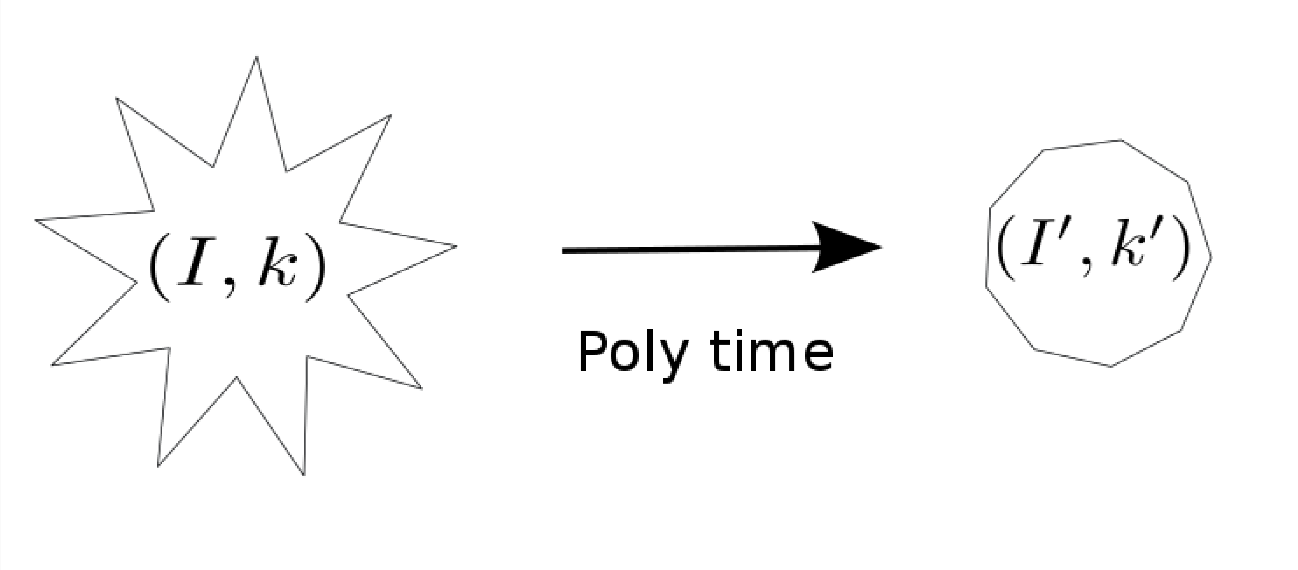
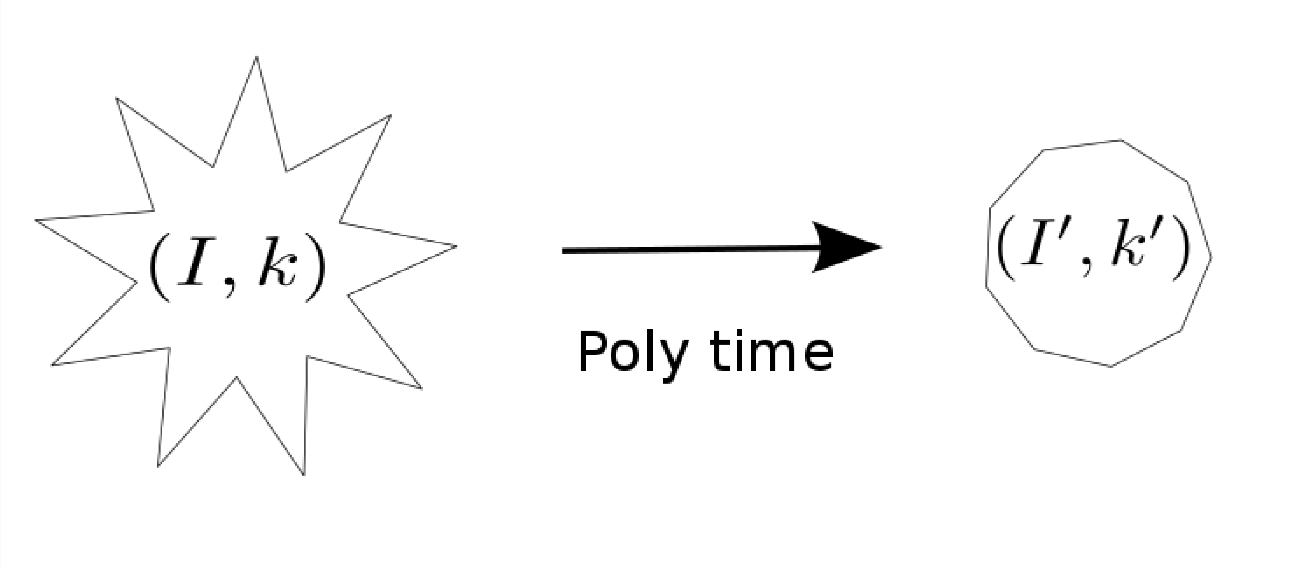
If S' is c-factor approximate solution to (G', k') then S is $\max\{c, \alpha\}$-factor solution to (G, k).

Lossy Kernelization

Kernelization



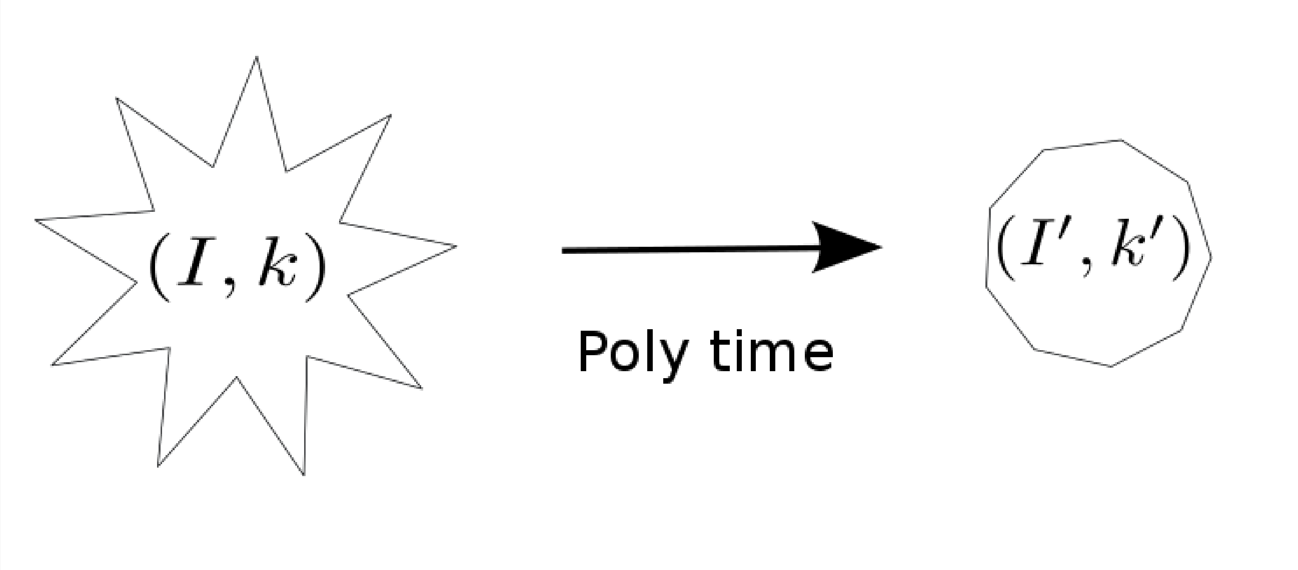
Kernelization



Parameterized problem Q admits a $h(k)$-kernel if there exists a poly-time algorithm $\mathcal{A}$ which given an input (I, k) outputs (I', k') such that

- $|I'| + k' \leq h(k)$
- (I, k) is YES instance iff (I', k') is YES instance

Kernelization



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How about optimization version?

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For a parameterized problem Q, its optimization analogue is a computable function

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Given instance I, parameter k and a solution S, the *value* of a solution S to an instance (I, k) of Q is $\Pi(I, k, S)$.

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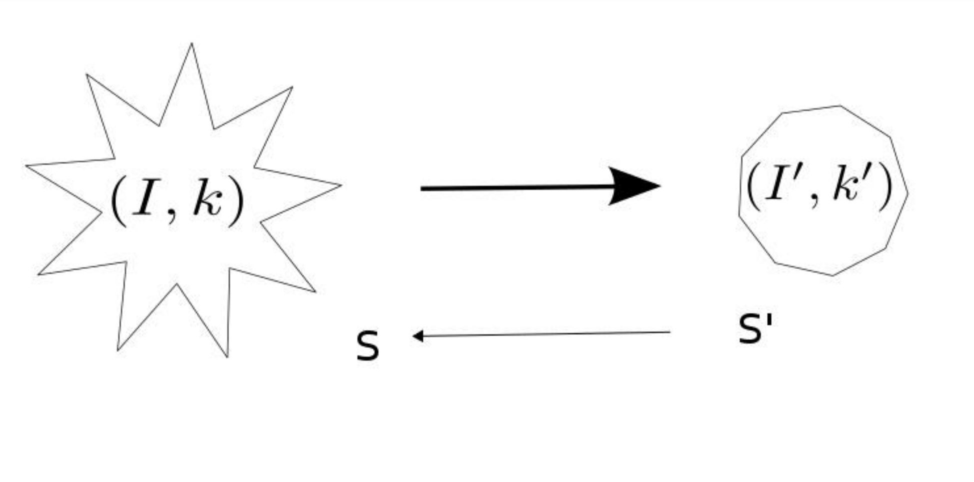
For parameterized minimization problems,

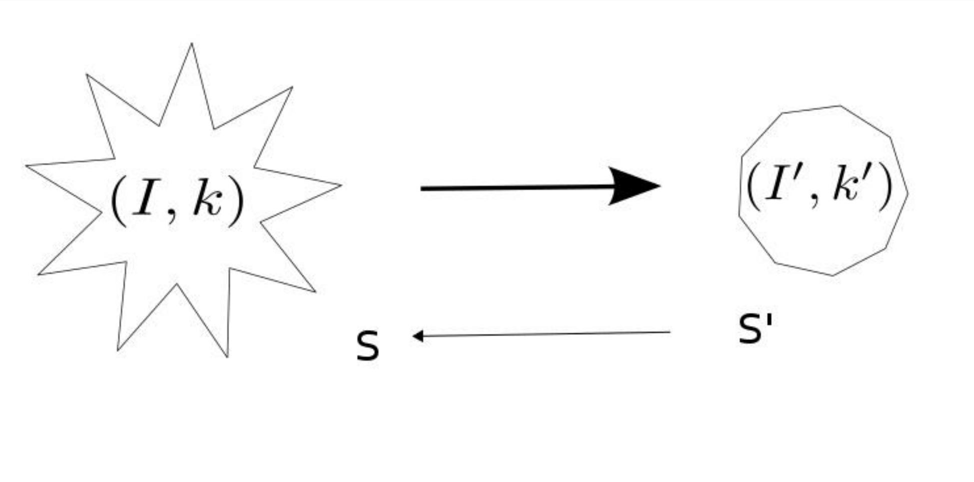
$$\mathrm{OPT}_\Pi(I, k) = \min_{S \in \Sigma^*; |S| \leq |I| + k} \{\Pi(I, k, S)\}$$

Lossy Kernelization



Given a solution S' to (I', k') can we construct a solution S to (I, k) which is as good as S'?

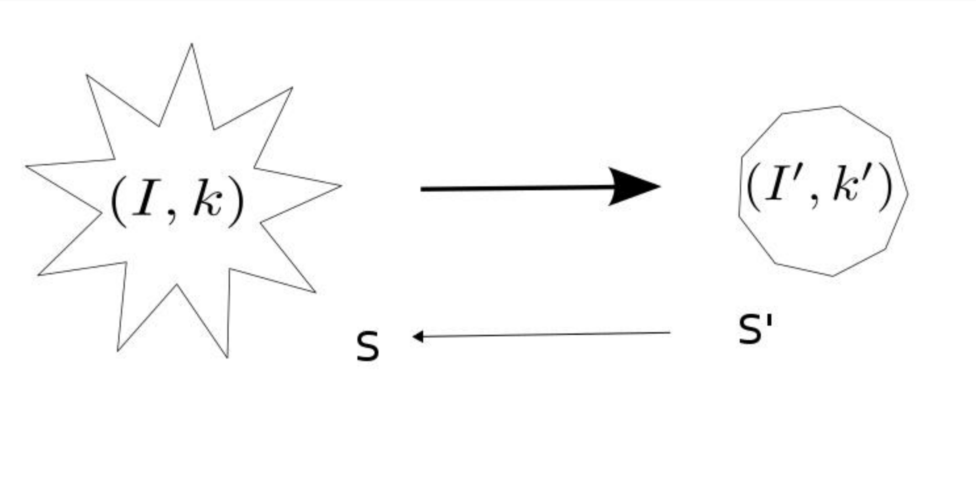
Lossy Kernelization



Given a solution S' to (I', k') can we construct a solution S to (I, k) which is as good as S'?

Quality of solution S' to (I', k') is $\frac{\Pi(I',k',S')}{\mathrm{OPT}(I',k')}$

Lossy Kernelization



Given (I', k', S') can we construct a solution S to (I, k) such that

$$\frac{\Pi(I,k,S)}{\mathrm{OPT}(I,k)} \leq \alpha \frac{\Pi(I',k',S')}{\mathrm{OPT}(I',k')}$$

for some constant α?

Lossy Kernelization

Definition (α-PTAS)

An *α-approximate polynomial-time preprocessing algorithm* (α-PTAS) is pair of two polynomial time algorithms as follows:

	Input	Output
Reduction Algorithm	(I, k)	(I', k')
Solution Lifting Algorithm	(I, k) and (I', k', S')	S

such that

$$\frac{\Pi(I, k, S)}{\mathrm{OPT}(I, k)} \leq \alpha \cdot \frac{\Pi(I', k', S')}{\mathrm{OPT}(I', k')}$$

Lossy Kernelization

Definition (Strict α-PTAS)

An *α-approximate polynomial-time preprocessing algorithm* (α-PTAS) is pair of two polynomial time algorithms as follows:

	Input	Output
Reduction Algorithm	(I, k)	(I', k')
Solution Lifting Algorithm	(I, k) and (I', k', S')	S

such that

$$\frac{\Pi(I, k, S)}{\mathrm{OPT}(I, k)} \leq max\{\alpha, \frac{\Pi(I', k', S')}{\mathrm{OPT}(I', k')}\}$$

Lossy Kernelization

Definition (Strict α-approximate kernel)

For a parameterized minimization problem Π if

1. Strict α-PTAS
2. the size of the output instance is upper bounded by a computable function $g : \mathbb{N} \to \mathbb{N}$ of k.

Lossy Kernelization

Minimization Problem

$$\Pi(I, k, S) = \begin{cases} \infty & \text{if } S \text{ is not a solution} \\ \min\{|S|, k+1\} & \text{otherwise} \end{cases}$$

Lossy Kernelization

Minimization Problem

$$\Pi(I, k, S) = \begin{cases} \infty & \text{if } S \text{ is not a solution} \\ \min\{|S|, k+1\} & \text{otherwise} \end{cases}$$

all solutions of size larger than $k + 1$ are equally bad.

Lossy Kernelization

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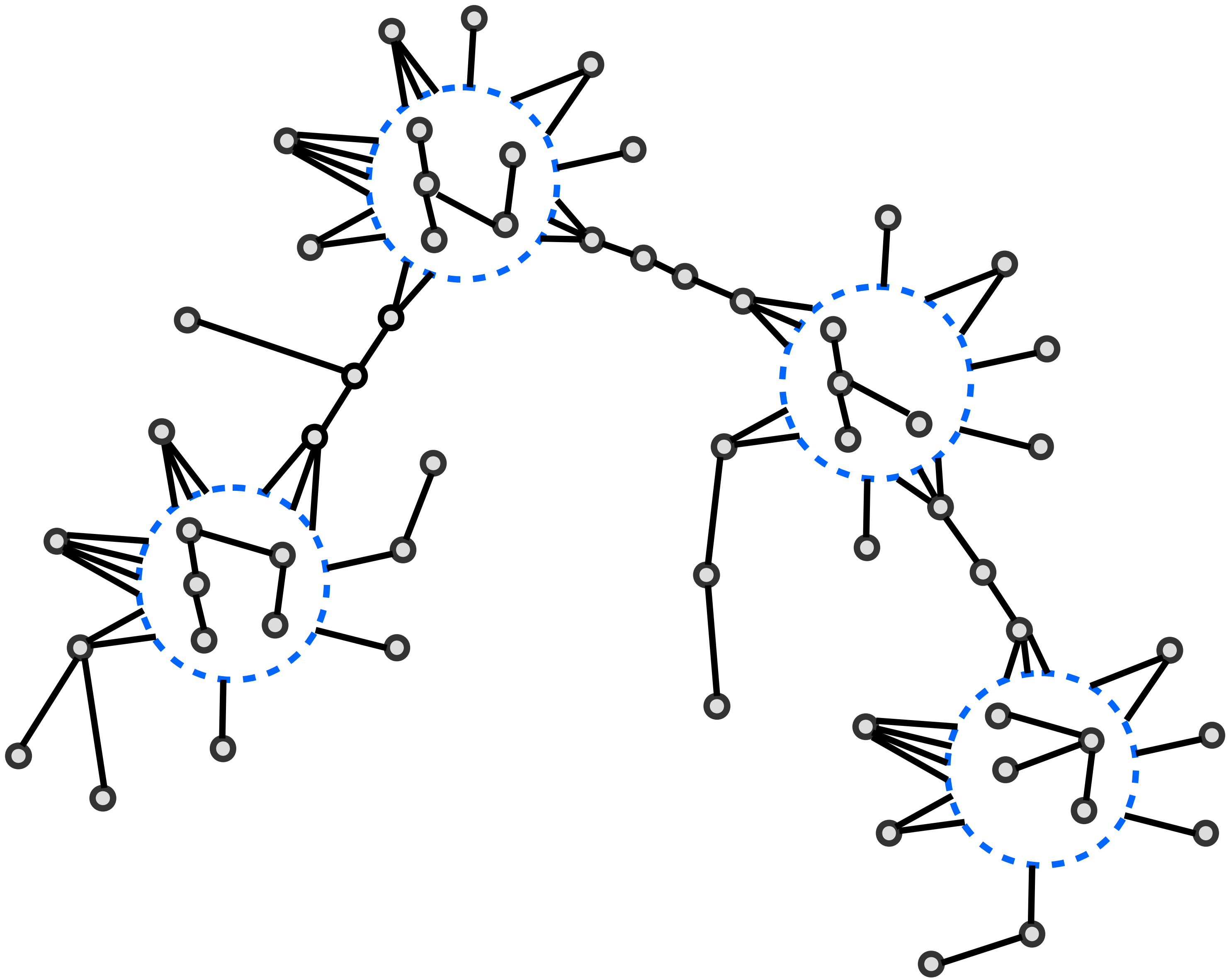
all solutions of size larger than $k+1$ are equally good.

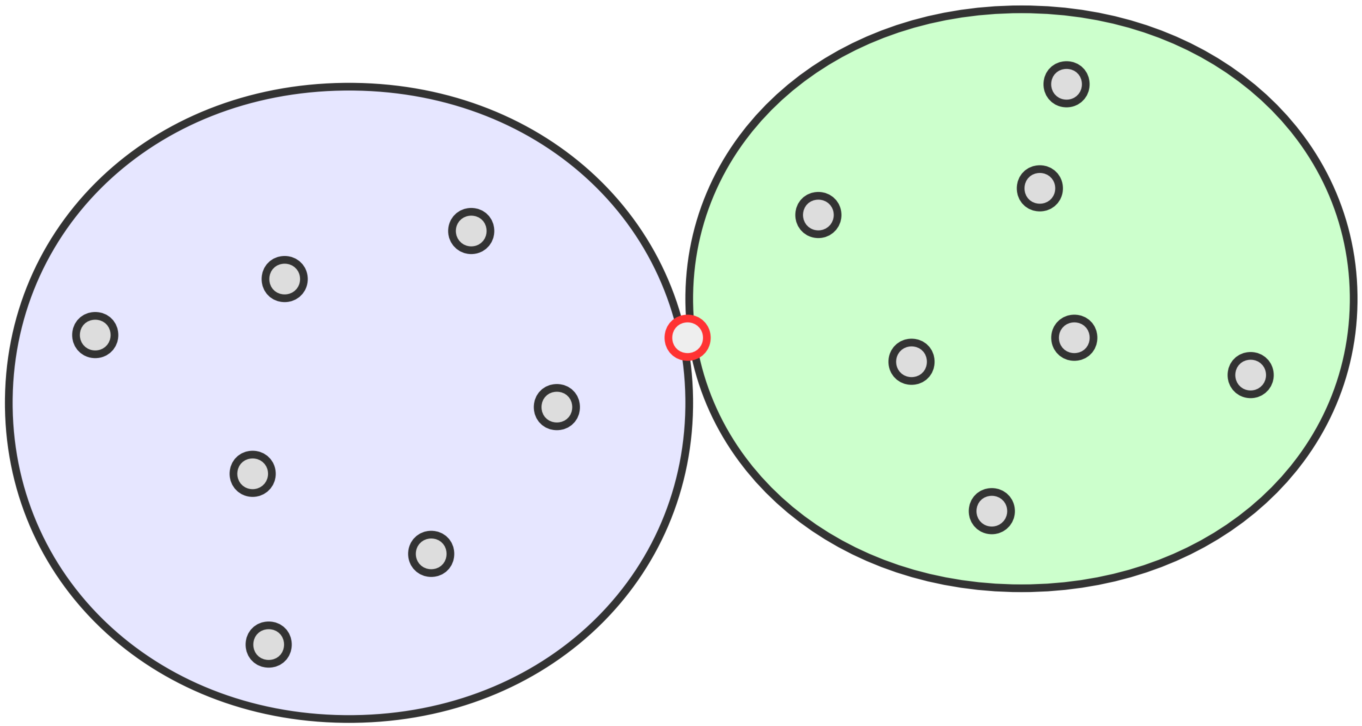
Lossy Kernel for Tree Contraction

Tree Contraction

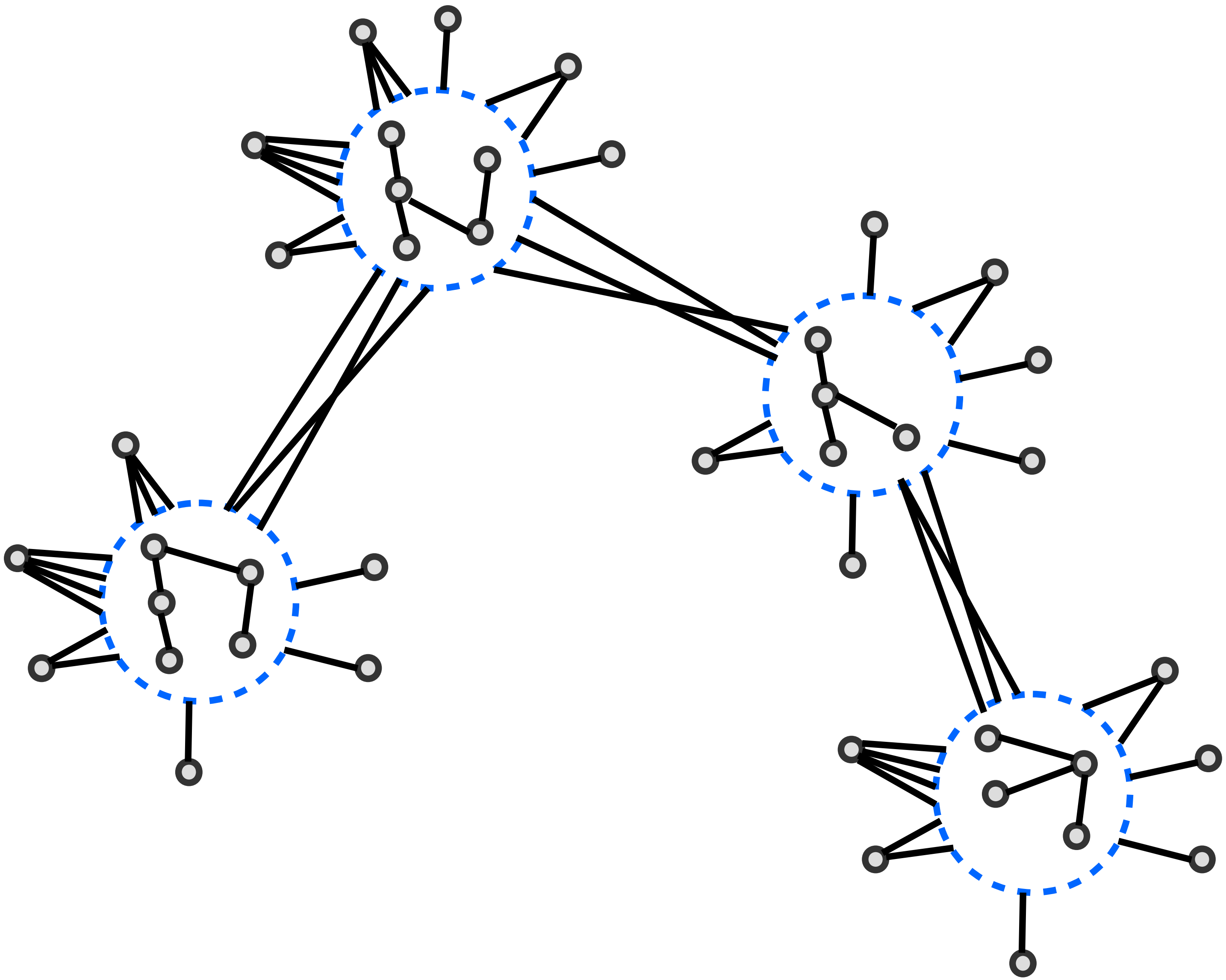


Tree Contraction

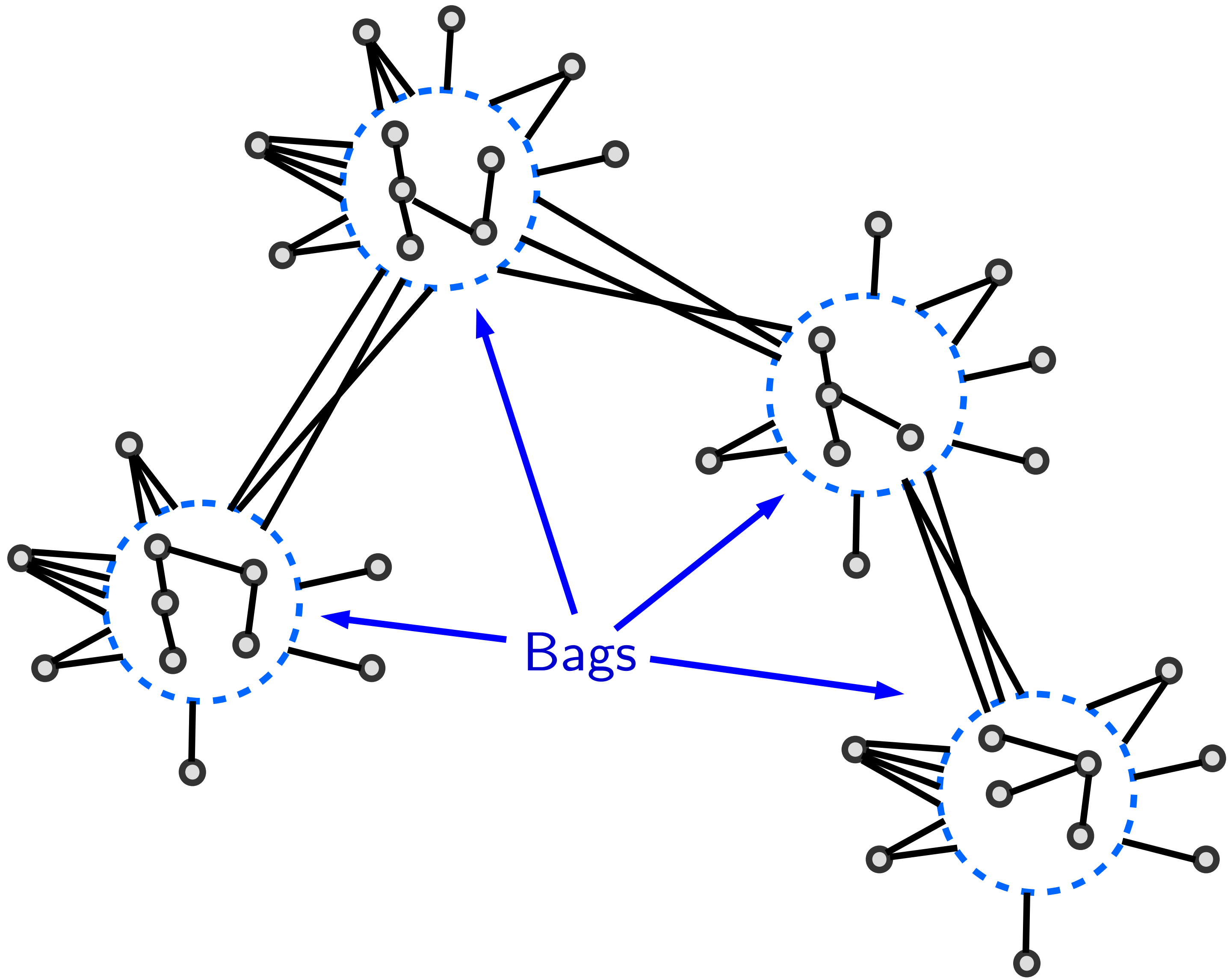
Consider each 2-vertex connected component separately



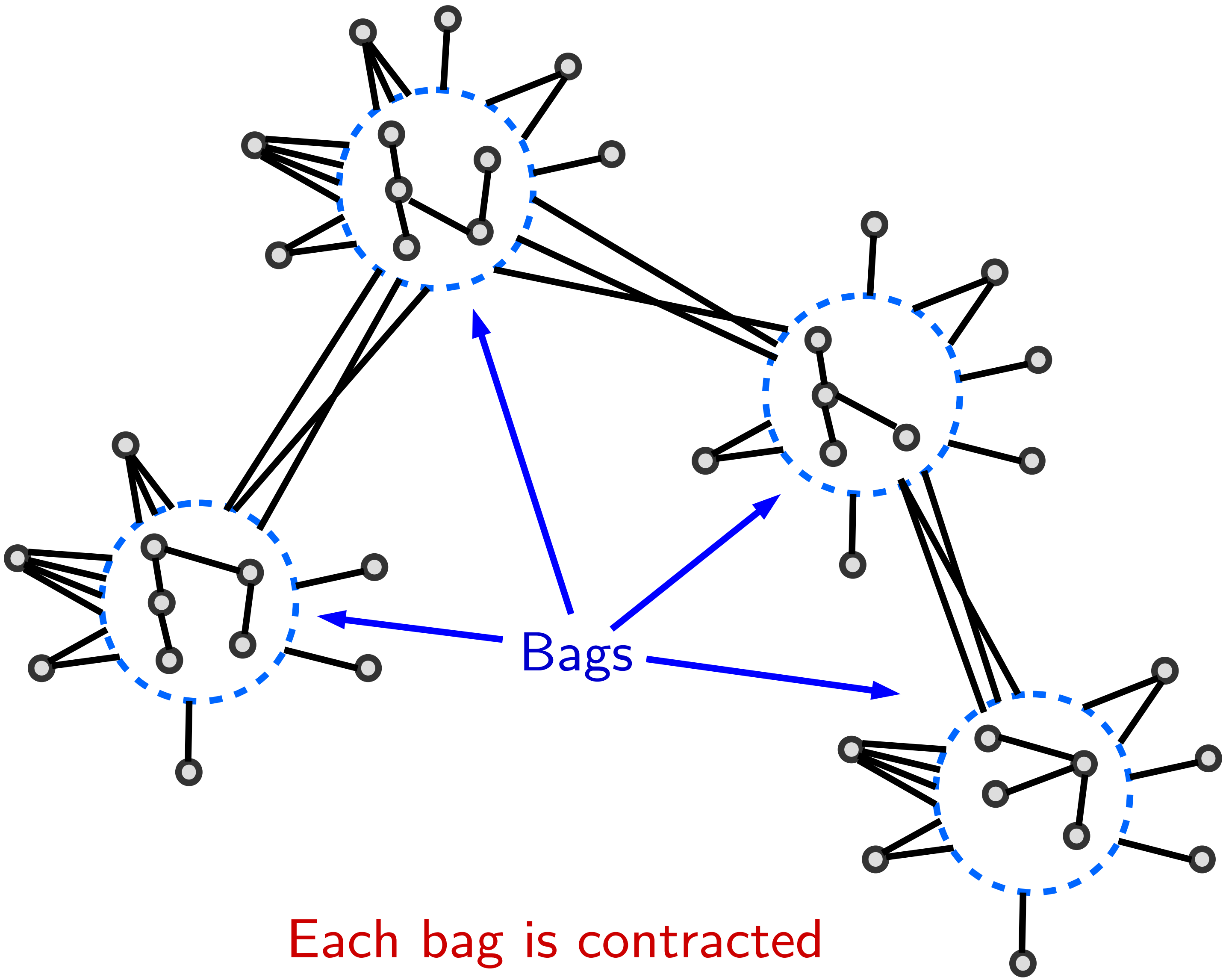
<u>Tree Contraction</u>



<u>Tree Contraction</u>

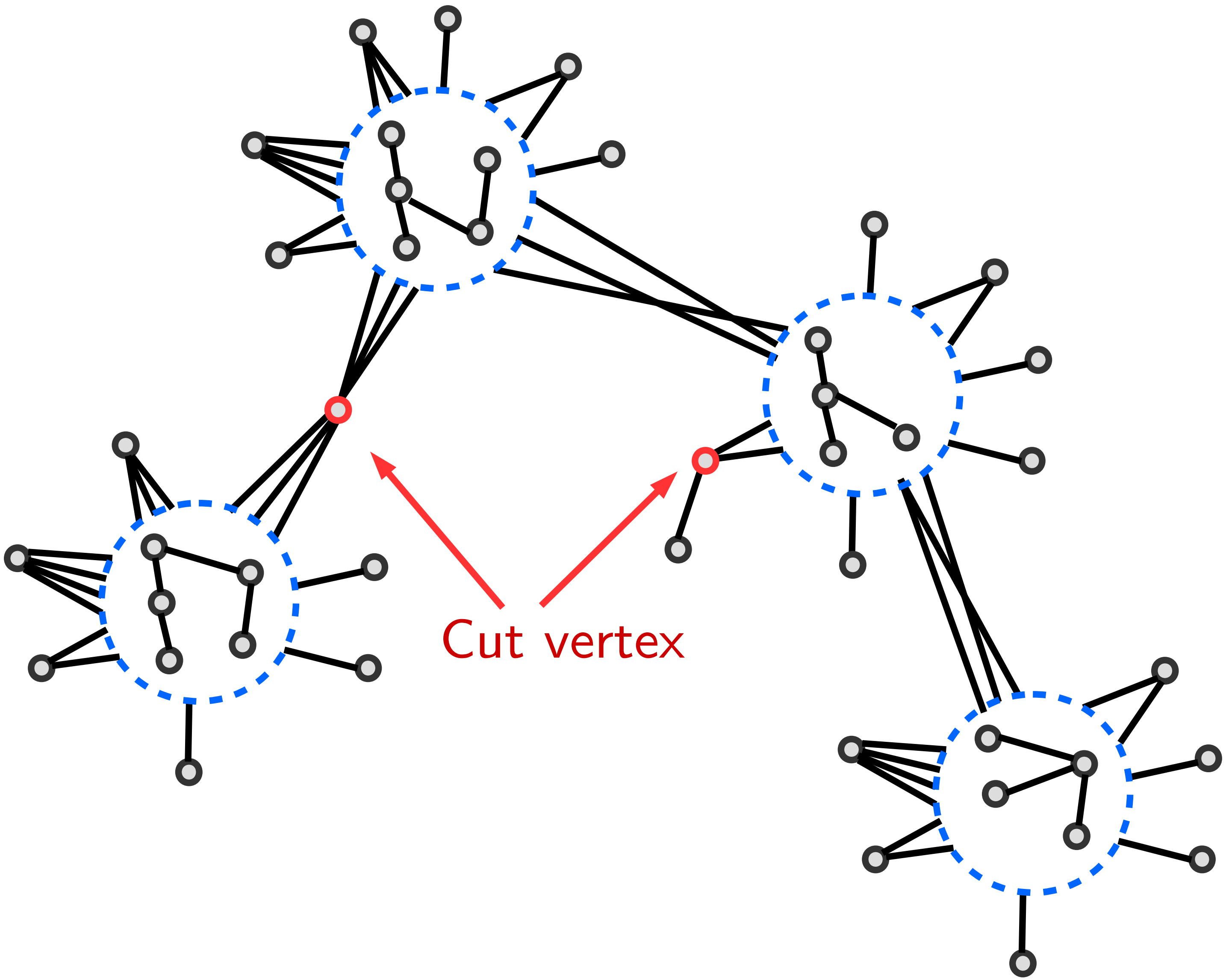


Tree Contraction

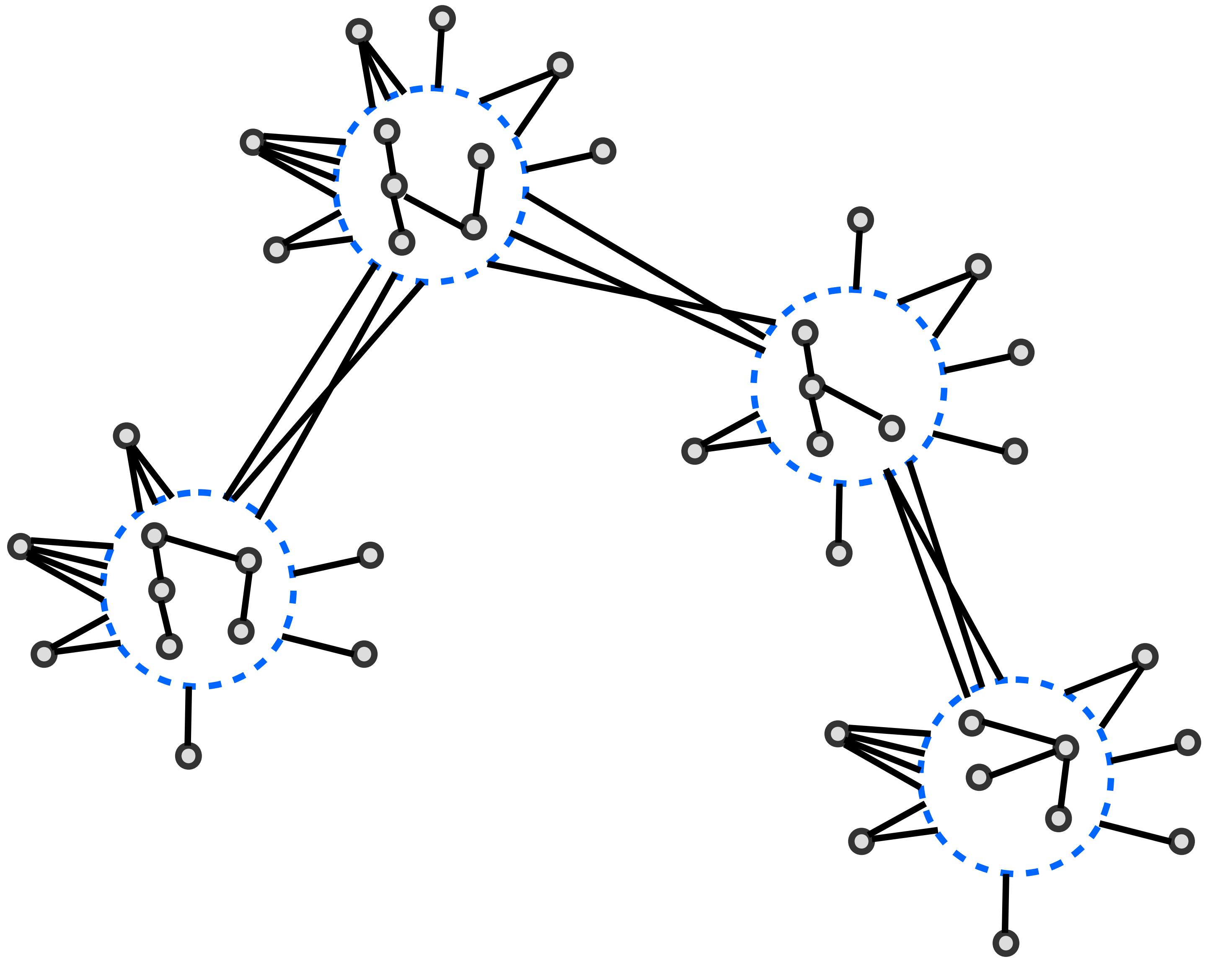


Each bag is contracted
to a distinct vertex

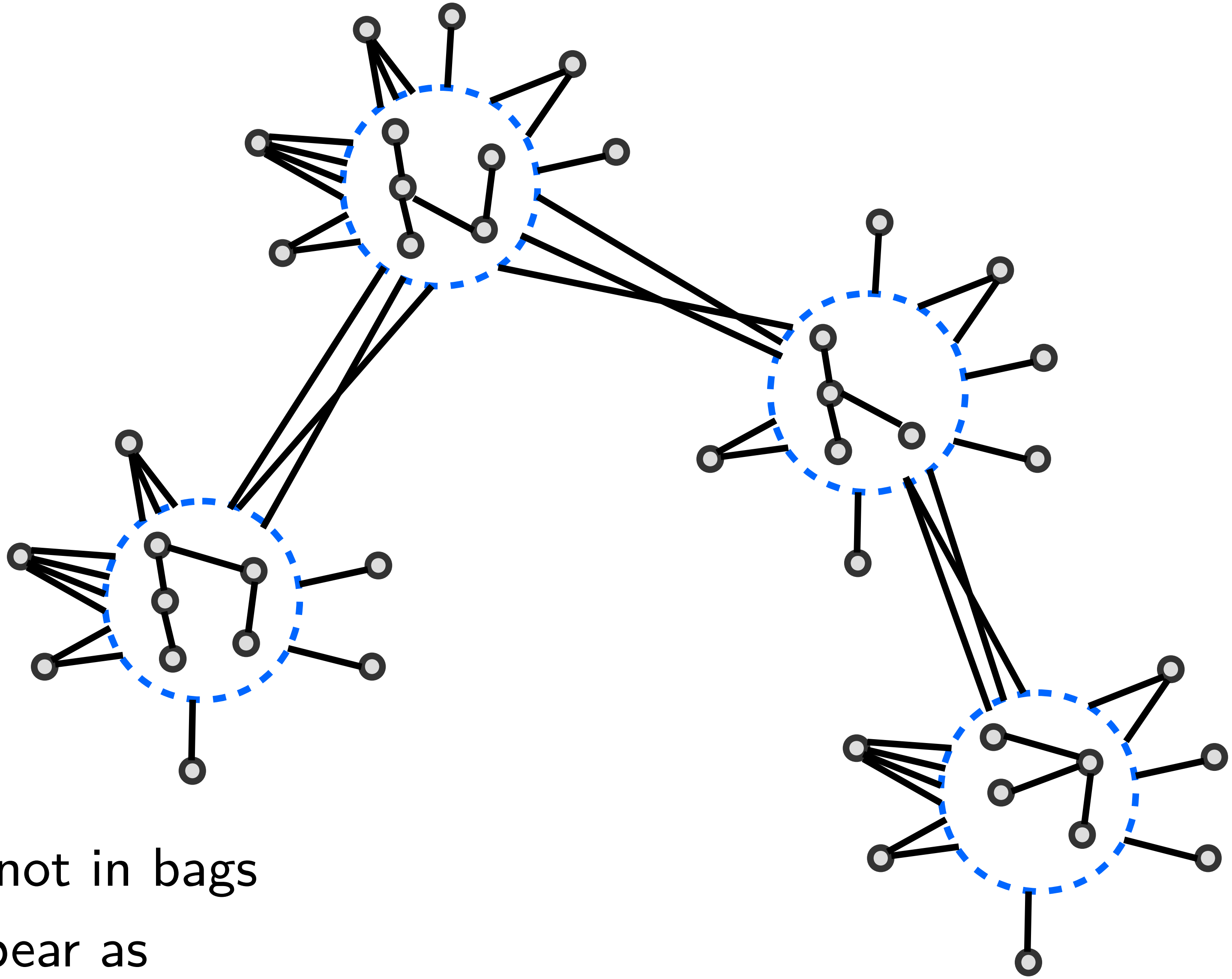
Tree Contraction



Tree Contraction



Tree Contraction

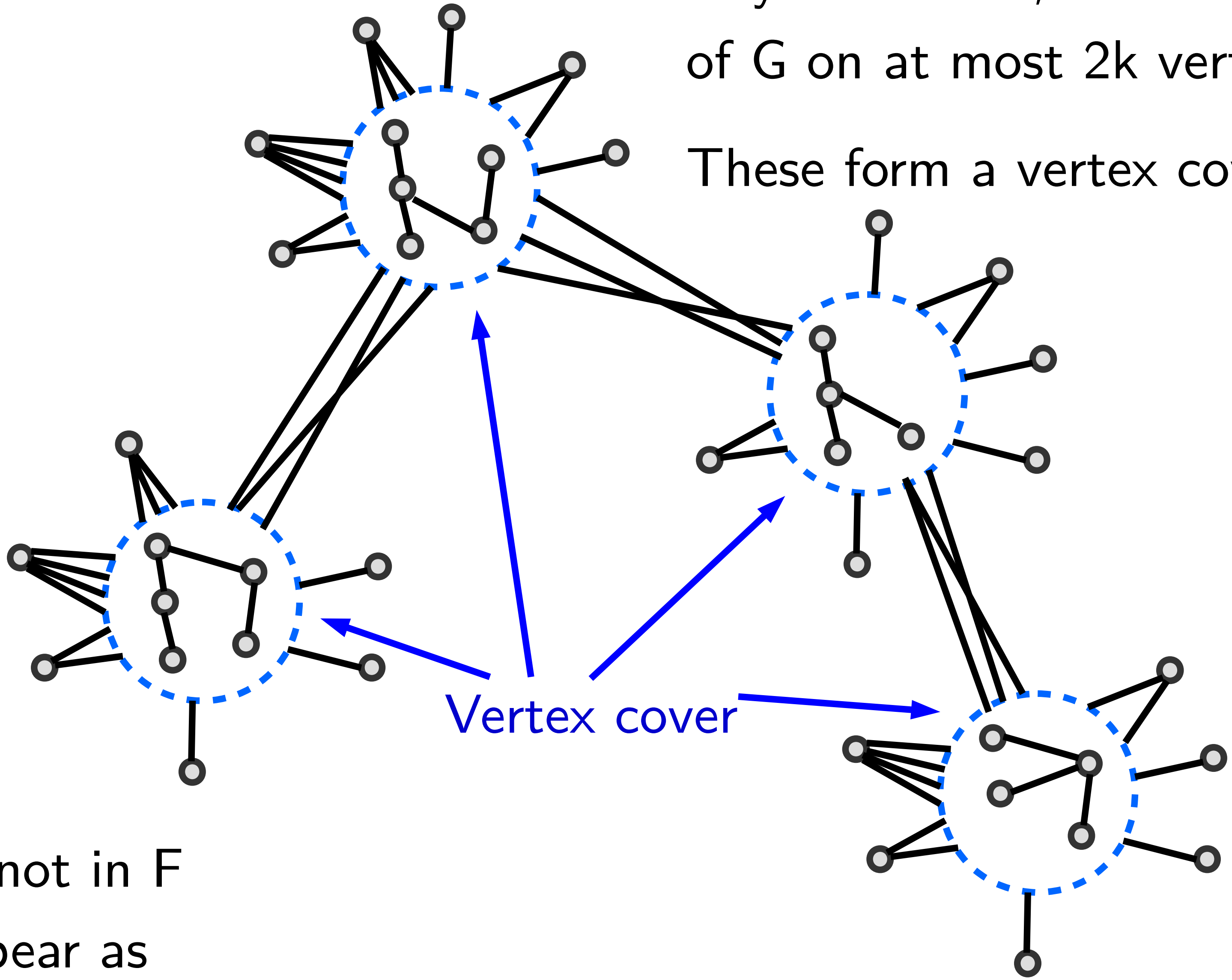


Vertices not in bags
must appear as
“leaves”

Tree Contraction

Any solution S, is a sub-forest of G on at most 2k vertices.

These form a vertex cover

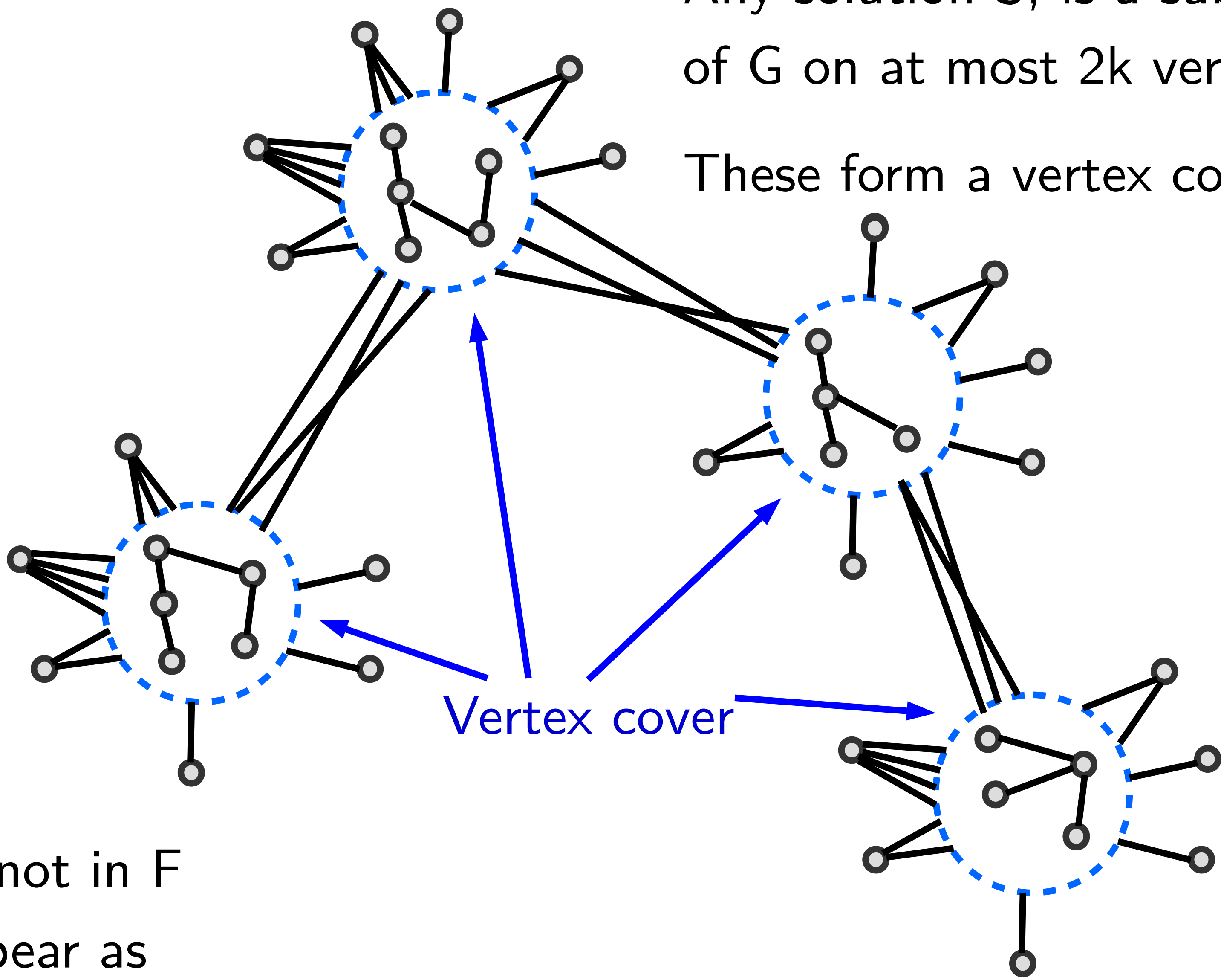


Vertices not in F must appear as "leaves"

Tree Contraction

Any solution S, is a sub-forest of G on at most 2k vertices.

These form a vertex cover



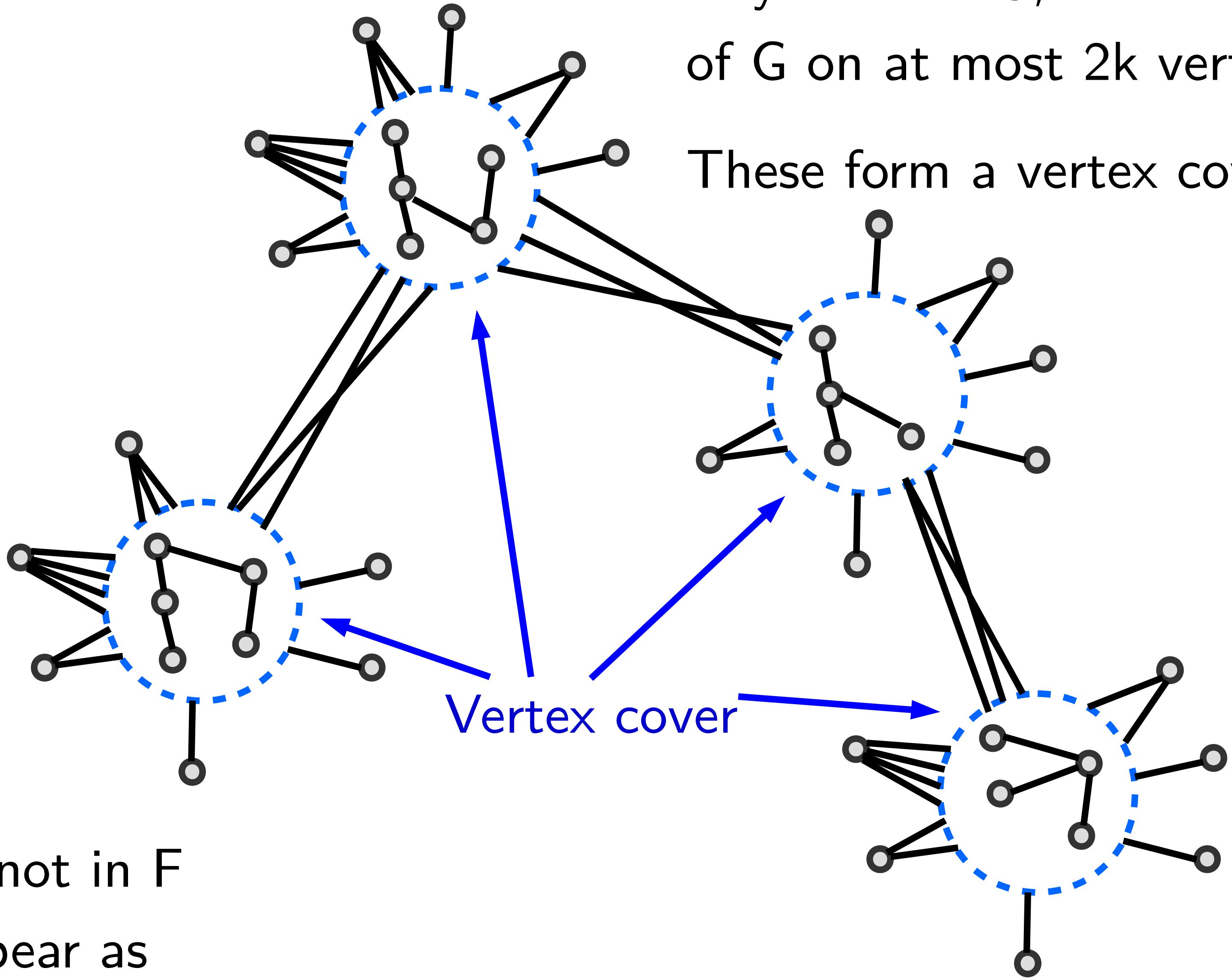
Vertices not in F must appear as "leaves"

If G is k-contractible to a tree, G has CVC of size 2k.

Tree Contraction

Any solution S, is a sub-forest of G on at most 2k vertices.

These form a vertex cover

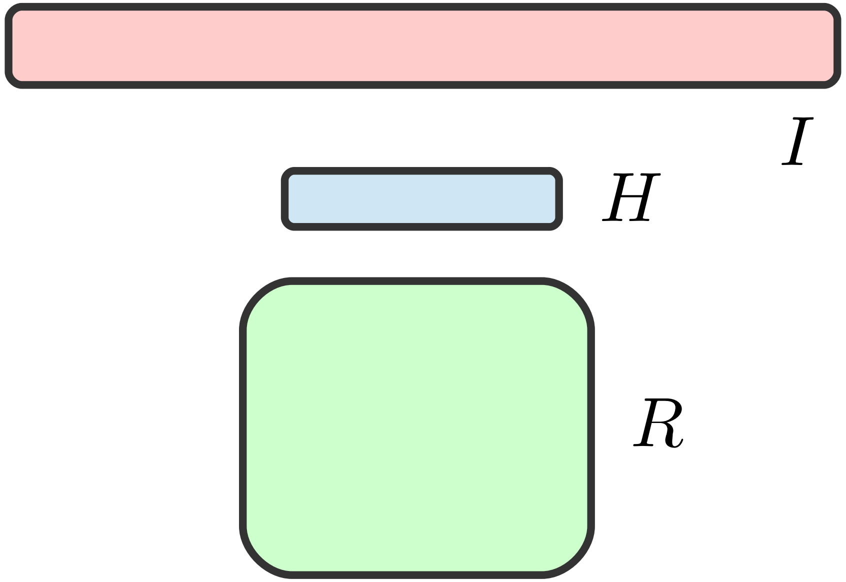


Vertices not in F must appear as "leaves"

Our goal is to reduce the leaves

Tree Contraction

- H : vertices of degree $\geq 2k+1$
- I : vertices whose neighborhood is contained in H
- R : the remaining vertices



Tree Contraction

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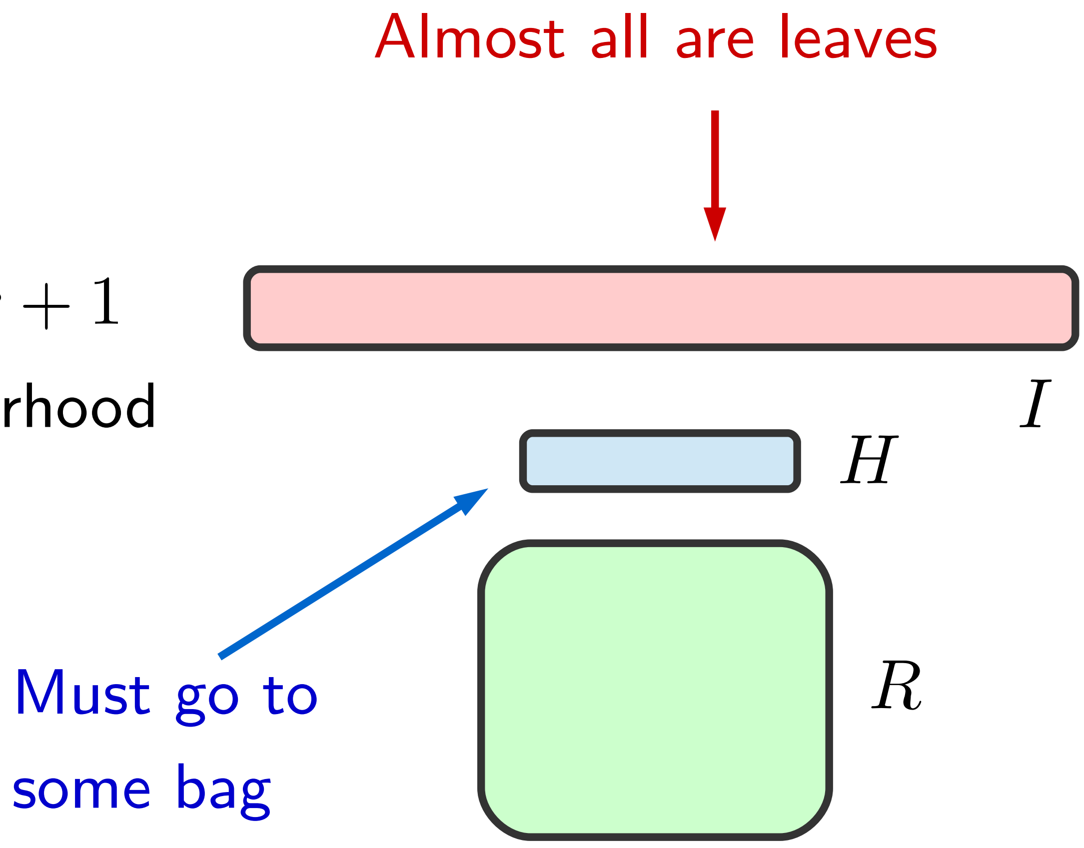

Tree Contraction

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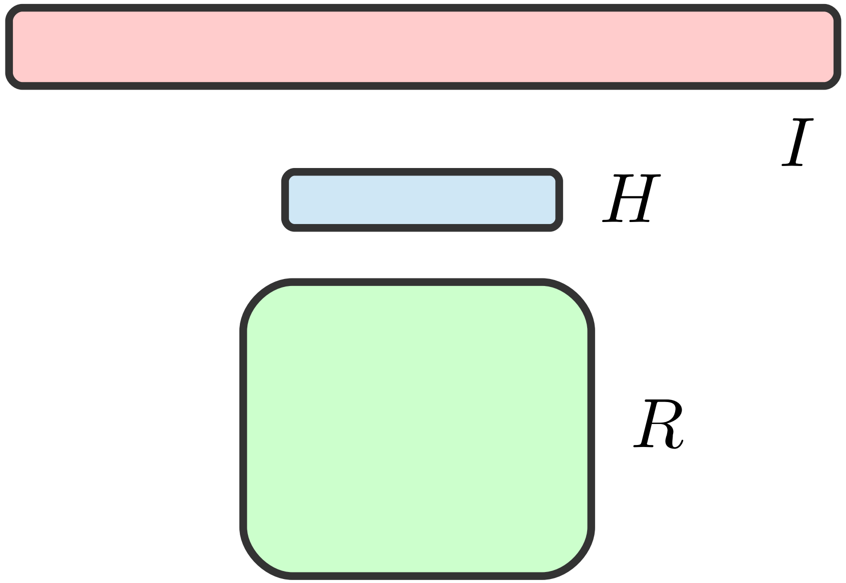
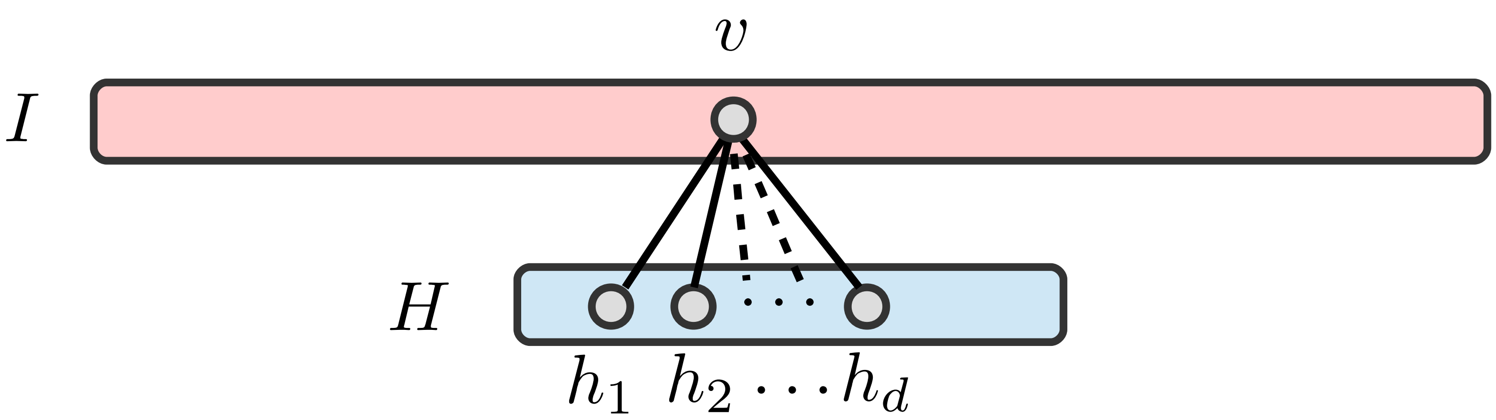


Goal is to reduce I

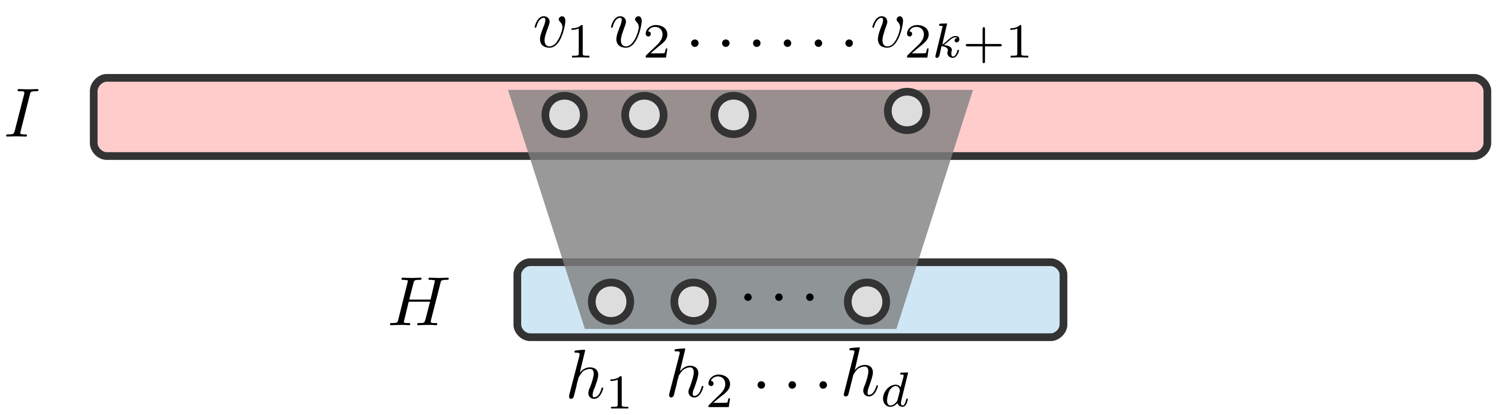
Tree Contraction



We don't know if $h_1, h_2, \dots, h_d$ are in the same bag

$$d = \lceil \frac{\alpha}{\alpha - 1} \rceil$$

Tree Contraction



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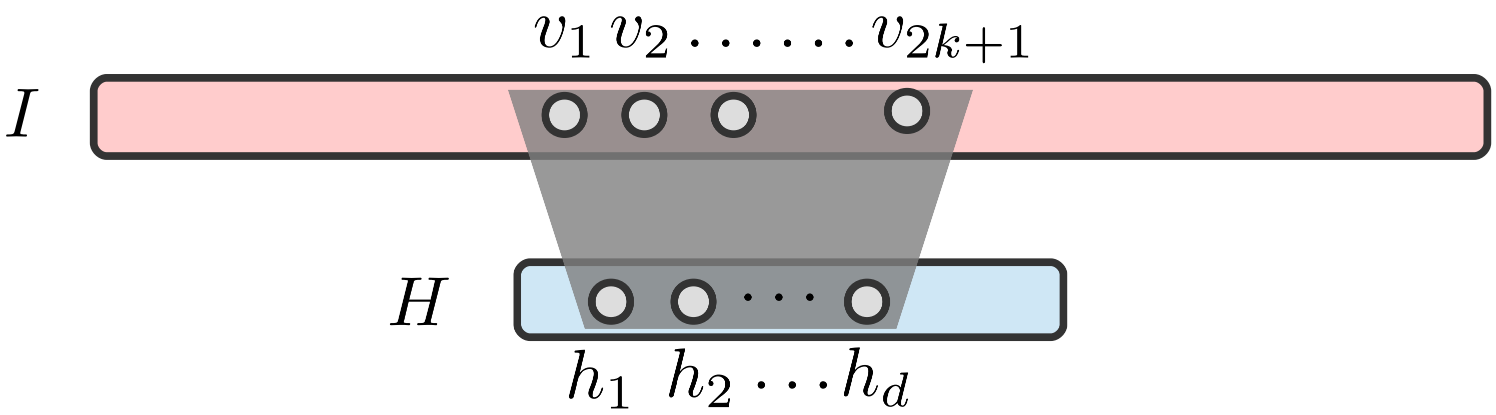
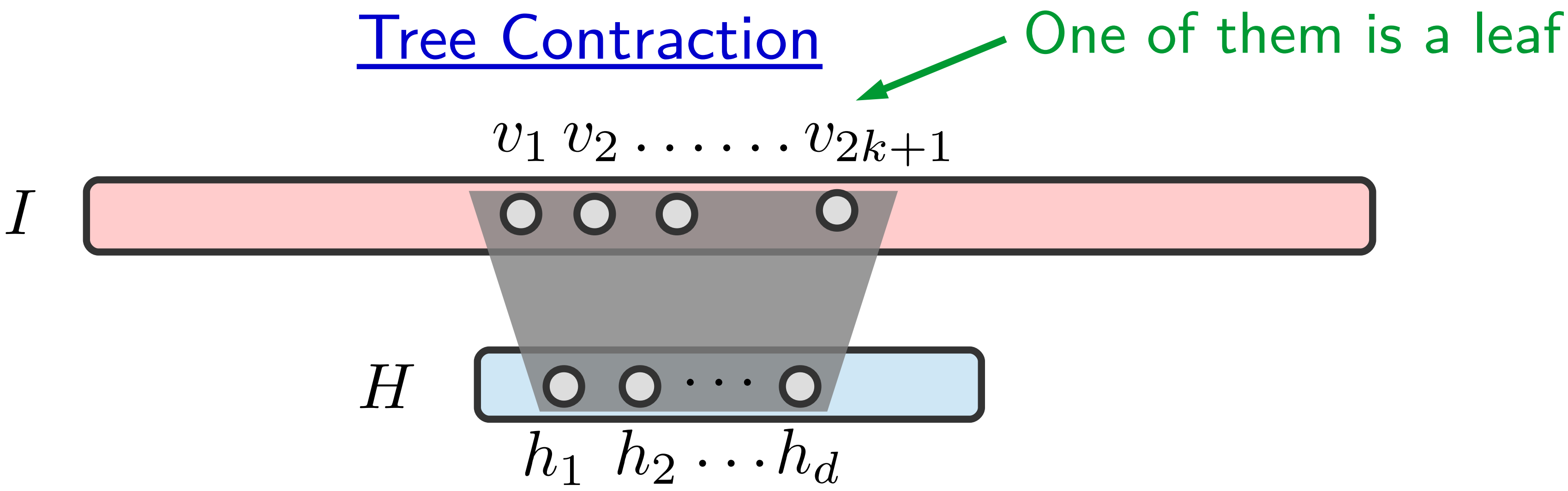
Tree Contraction



Claim: $h_1, h_2, \ldots, h_d$ must all go to the same bag

$$d = \lceil \frac{\alpha}{\alpha - 1} \rceil$$

Tree Contraction



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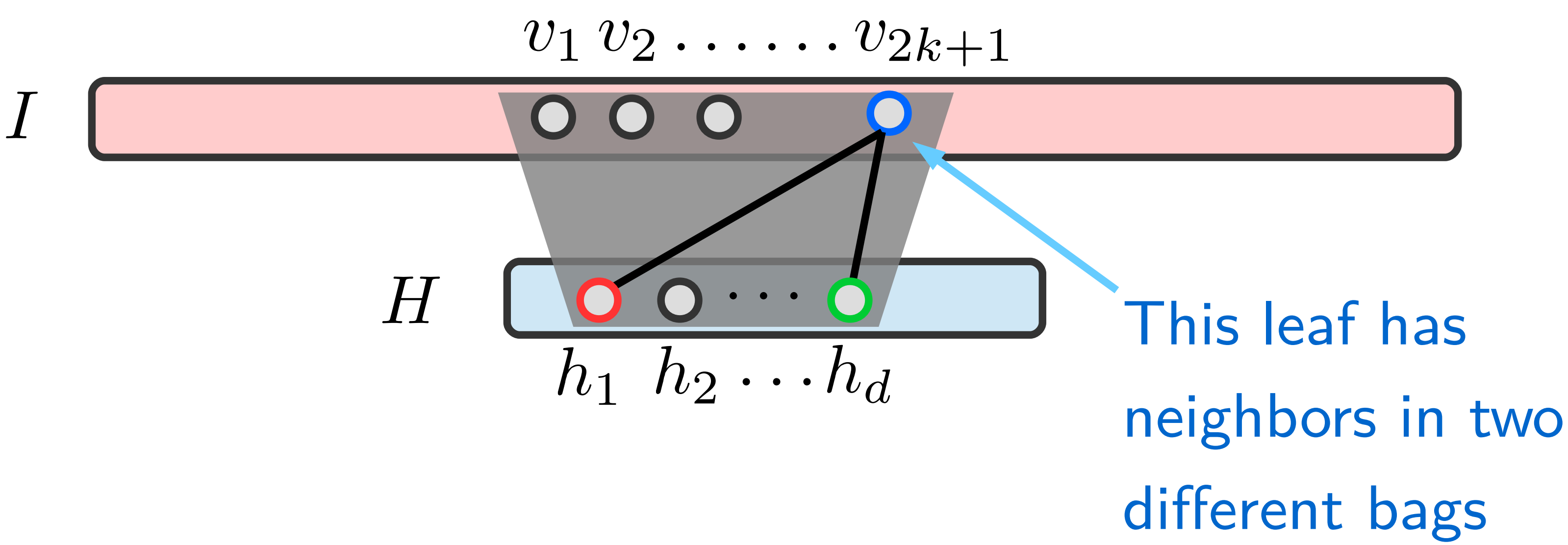
Tree Contraction



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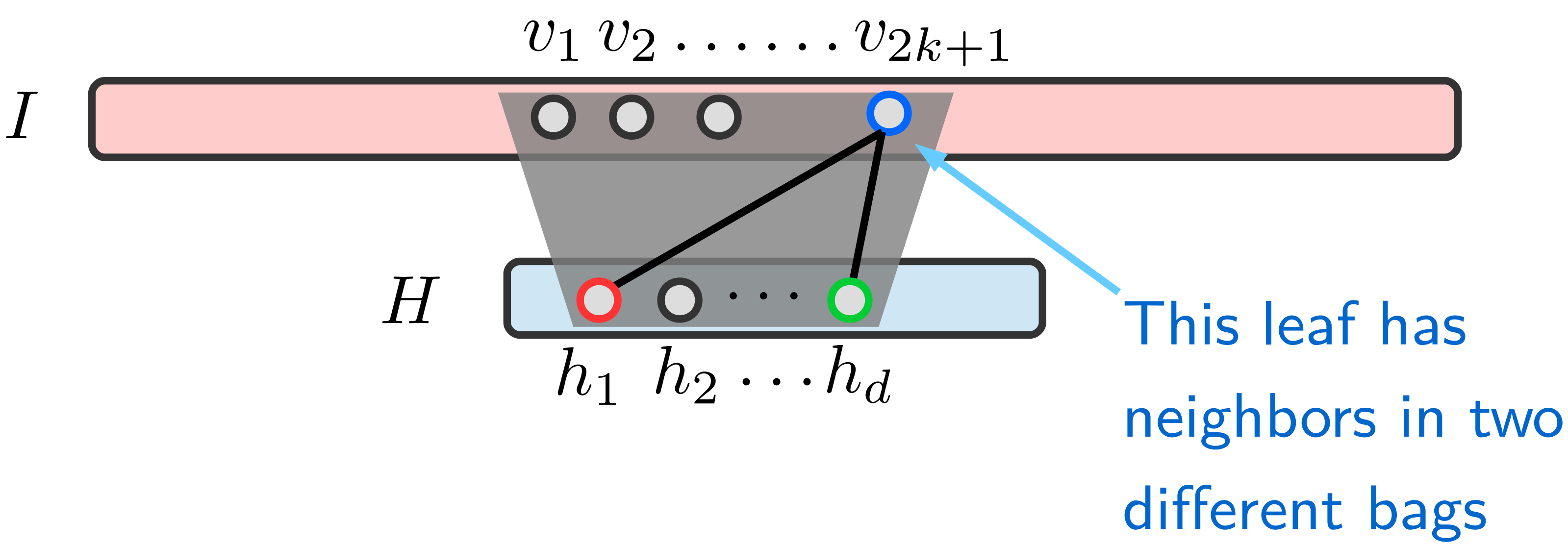
Tree Contraction



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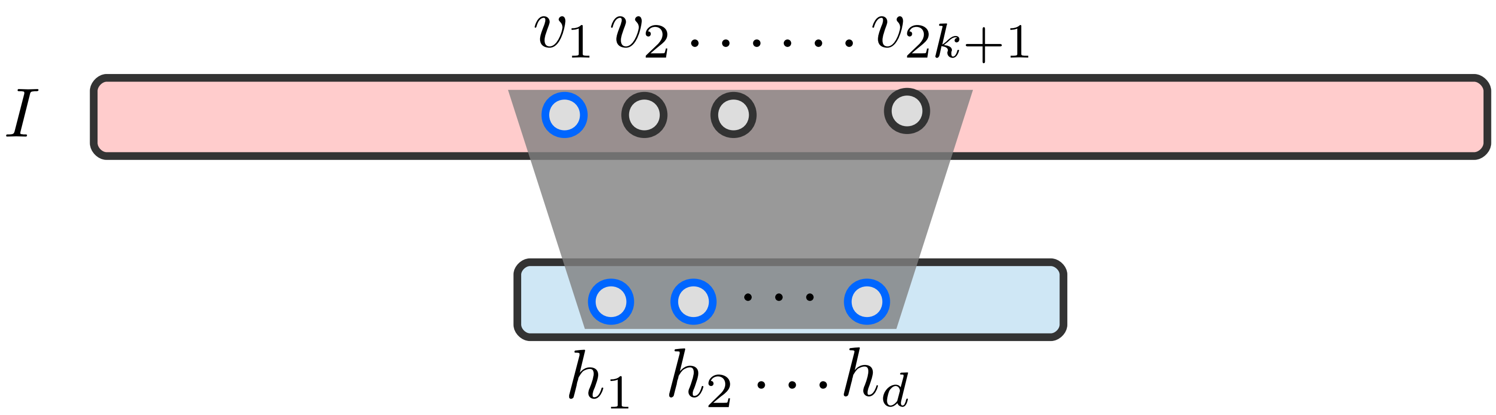
Tree Contraction



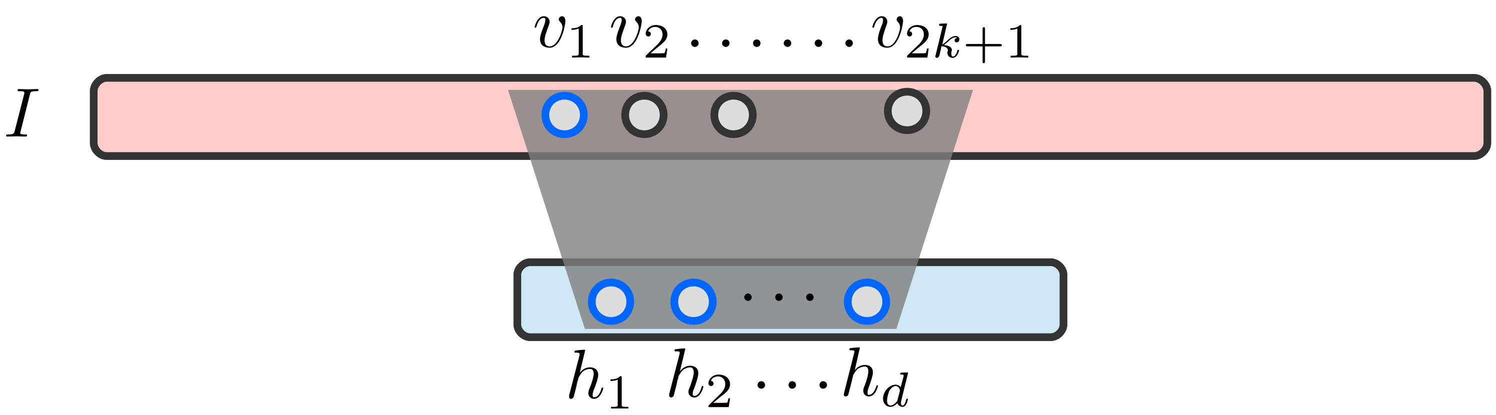
Claim: $h_1, h_2, \dots, h_d$ must all go to the same bag

Reduction Rule :

Contract $\{v_1, h_1, h_2, \dots, h_d\}$ to a vertex in H

$$d = \lceil \frac{\alpha}{\alpha - 1} \rceil$$

Tree Contraction



Claim: $h_1, h_2, \ldots, h_d$ must all go to the same bag

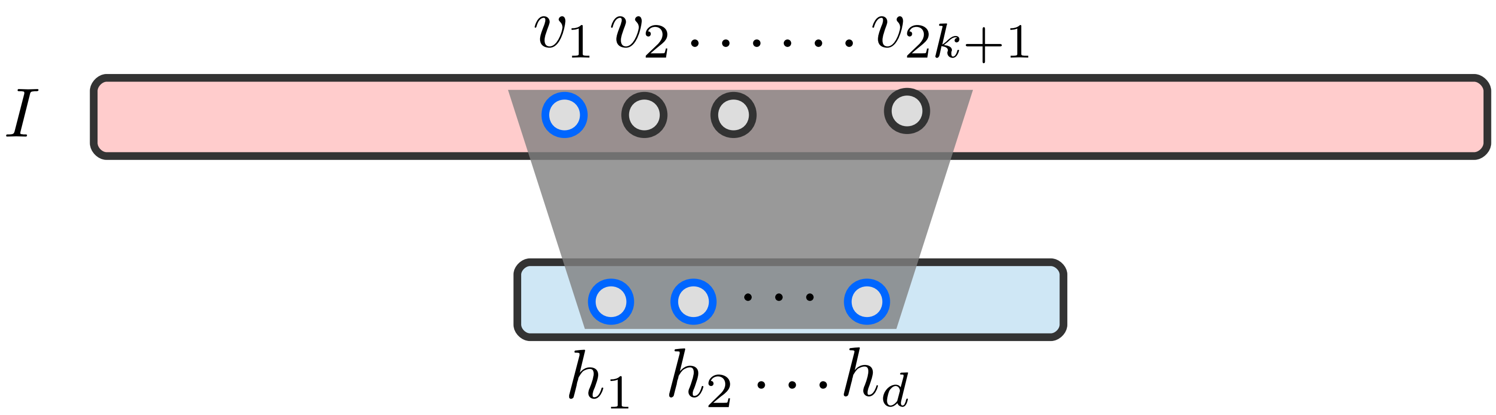
Reduction Rule :

Contract $\{v_1, h_1, h_2, \ldots, h_d\}$ to a vertex in H

This is α-safe

$$d = \lceil \frac{\alpha}{\alpha - 1} \rceil$$

Tree Contraction



Claim: $h_1, h_2, \ldots, h_d$ must all go to the same bag

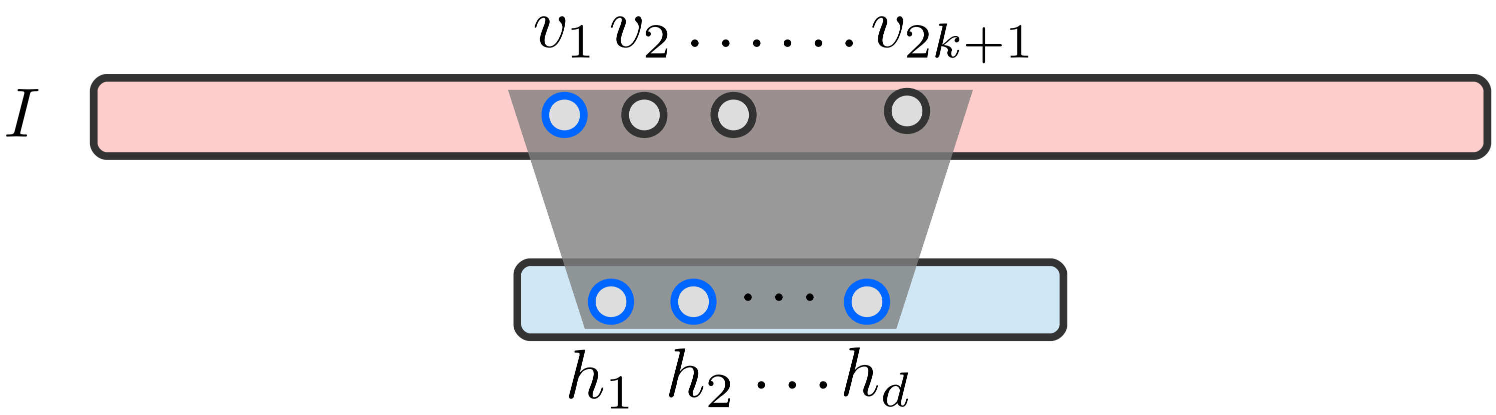
Reduction Rule :

Contract $\{v_1, h_1, h_2, \ldots, h_d\}$ to a vertex in H

We argue that $|I| \leq \mathcal{O}(k^d)$

$$d = \lceil \frac{\alpha}{\alpha - 1} \rceil$$

Tree Contraction



Claim: $h_1, h_2, \ldots, h_d$ must all go to the same bag

Reduction Rule :

Contract $\{v_1, h_1, h_2, \ldots, h_d\}$ to a vertex in H

We get a lossy kernel for Tree Contraction of size $\mathcal{O}(k^d)$

Thank you!

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