Harmonious Coloring: Parameterized and Exact Algorithms

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Outline

- Harmonious Coloring
- Parameterized Complexity
- FPT Algorithms
- Exact Algorithms
- Open Problems

Introduction

PROPER-COLORING
Input: A graph G
Question: Find minimum int k such that graph G can be partitioned into k independent sets? (such that there is at least one edge between any two partitions)

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HARMONIOUS-COLORING
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- NP-complete on general graphs [4]
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- No exact algorithm is known which is better than $2^{\mathcal{O}(n \log n)}$
- Polynomial time algorithms for tree of bounded degree, path, cycles, grids [3].

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VERTEX COVER(G, k)	$\mathcal{O}(2^k \cdot n^2)$
FEEDBACK VERTEX SET(G, k)	$\mathcal{O}(3.6181^k \cdot n^c)$
INDEPENDENT SET(G, k)	No $f(k) \cdot \|I\|^{\mathcal{O}(1)}$ algorithm
PROPER COLORING(G, k)	No $f(k) \cdot \|I\|^{\mathcal{O}(1)}$ algorithm

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Independent Set(G, k)	No such $h(k)$ exists
Proper Coloring(G, k)	No such $h(k)$ exists

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Theorem

A parameterized problem Q is FPT iff it admits a kernelization algorithm.

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  - vertex cover (*vc*): minimum # vertices needs to be deleted to obtain independent set

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HARMONIOUS COLORING **Parameter:** k
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Theorem

HARMONIOUS COLORING *admits a kernel with $\mathcal{O}(k^2)$ vertices and $\mathcal{O}(k^2)$ edges.*

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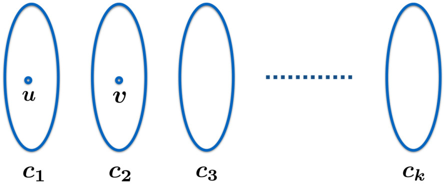
VC-HARMONIOUS COLORING **Parameter:** $|X|$
Input: A graph G, a vertex cover X of G, an int k
Question: Is there a harmonious coloring of G with k colors?

FPT algorithms for Harmonious Coloring

Any Harmonious coloring of graph G



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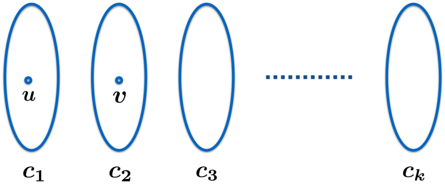
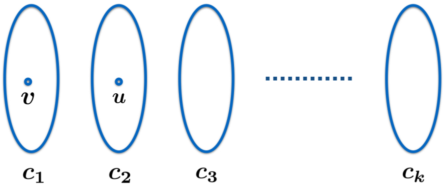


If $N(u) = N(v)$ then following is also a valid Harmonious coloring



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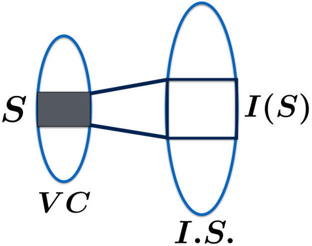

For every $S \subseteq X$, define $I(S) = \{u \in I | N(u) = S\}$

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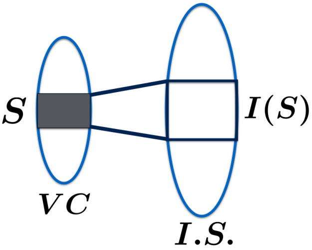

For every $S \subseteq X$, define $I(S) = \{u \in I | N(u) = S\}$

Observation

Every vertex in $I(S)$ is identical with respect to any harmonious coloring

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Observation

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The brand of a vertex v in I with respect to X is the set $N(v)$.

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Definition (type)

A type Z with respect to X is a $\ell + 1$ sized tuple where the first entry is subset of X of cardinality at most 1, and each of the remaining ℓ entries is either $\emptyset$ or a distinct brand of a vertex in I.

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A type $Z \equiv (Y; S_1, S_2, \ldots, S_\ell)$. $\#$ types $\leq \ell \cdot \binom{2^\ell}{\ell} \leq \ell \cdot 2^{\ell^2}$.

FPT algorithms for Harmonious Coloring

Definition (Color Class of type Z)

Color class C of harmonious coloring h is of type $Z = (Y; S_1, S_2, \ldots, S_\ell)$ if $C \cap X = Y$ and for every $u \in C \cap I$ there exists S_i in type Z such that $\text{brand}(u) = S_i$.

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$\mathcal{Z}$ be the collection of all *compatible* color classes

FPT algorithms for Harmonious Coloring

Objective function

$$\text{minimize} \sum_{i=1}^{|\mathcal{Z}'|} z_i$$

we encode the aim of minimizing number of color classes used.

FPT algorithms for HARMONIOUS COLORING

For every $S \subseteq X$ and $j \in [|\mathcal{Z}'|]$ define

$$b_j^S = 1 \text{ if there is brand } S \text{ in type } Z_j; \text{ otherwise } 0$$

There are at most $|I(S)|$ many vertices of brand S.

$$\sum_{j=1}^{|\mathcal{Z}'|} z_j \cdot b_j^S = |I(S)| \qquad \forall S \subseteq X \tag{1}$$

FPT algorithms for Harmonious Coloring

For every $x \in X$ and $j \in [|\mathcal{Z}'|]$ define

$$c_j^x = 1 \text{ if } \{x\} \text{ is the first entry in type } Z_j; \text{ otherwise } 0$$

There can be at most one color class which contains vertex x in X.

$$\sum_{j=1}^{|\mathcal{Z}'|} z_j \cdot c_j^x = 1 \qquad \forall x \in X \tag{2}$$

FPT algorithms for Harmonious Coloring

$$\text{minimize} \sum_{i=1}^{|\mathcal{Z}'|} z_i$$

such that

$$\sum_{j=1}^{|\mathcal{Z}'|} z_j \cdot b_j^S = |I(S)| \qquad \forall S \subseteq X \tag{3}$$

$$\sum_{j=1}^{|\mathcal{Z}'|} z_j \cdot c_j^x = 1 \qquad \forall x \in X \tag{4}$$

FPT algorithms for Harmonious Coloring

Theorem

An Integer Linear Programming *instance of size L with p variables can be solved using*

$$(p^{2.5p+o(p)} \cdot (L + \log M_x) \cdot \log(M_x \cdot M_c))$$

arithmetic operations and space polynomial in $L + \log M_x$, where M_x is an upper bound on the absolute value a variable can take in a solution, and M_c is the largest absolute value of a coefficient in the vector c.

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In previous ILP, # variables is $\leq \ell \cdot 2^{\ell^2}$; # constraints $2^\ell + \ell$; $M_x = n$ and $M_c = 1$

FPT algorithms for Harmonious Coloring

Theorem

Harmonious Coloring, *parameterized by the size of a vertex cover of the input graph, is fixed-parameter tractable.*

Exact Algorithms

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For $u, v \in V(G)$, if $d(u, v) = 2$ they can't be in the same color class.

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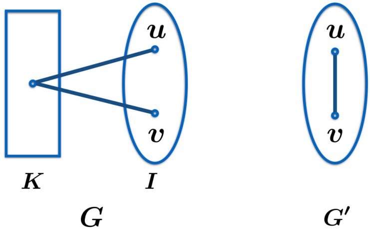
For given instance (G, k), concentrate on coloring I with $k - |K|$ many colors.

Exact Algorithms

Construct an auxiliary graph G'



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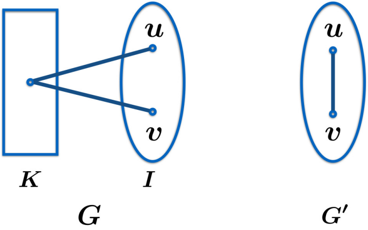

Lemma

ϕ is Harmonious coloring of G iff ϕ_I is proper coloring of G'.

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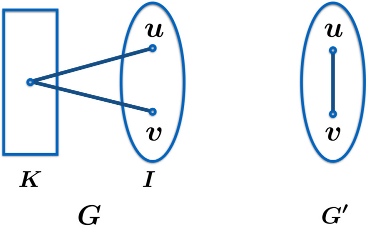

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Theorem

HARMONIOUS COLORING *on Split graphs can be solved in $\mathcal{O}(2^n)$ time.*

Other Variations of Coloring

q-PROPER	Partition into q-independent sets viz color classes	Minimization
HARMONIOUS	At most one edge between any two color classes.	Minimization
ACHROMATIC	At least one edge between any two color classes.	Maximization
b-CHROMATIC	Each color class there is a vertex that has a neighbor in every other color class.	Maximization
GRUNDY	Worst performance of greedy algorithm.	Maximization

Open Problems

Coloring	tw	FVS	VC
q-Proper Coloring	$\mathcal{O}(k^{tw})$	FPT	$2^{\mathcal{O}(k)}$
Harmonious Coloring	para NP-hard	para NP-hard	FPT
Achromatic Coloring	para NP-hard	para NP-hard	Open
b-Chromatic Coloring	Open	Open	Open
Grundy Coloring	Open	Open	FPT

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Thank you!