Exact and Parameterized Algorithms for (k, i)-coloring

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Outline

1. Graph Coloring
2. Introduction and Motivation
3. Exact Algorithms
4. Parameterized Algorithms
5. Conclusions

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Goal: Find minimum integer q such that graph $V(G)$ can be partitioned into q independent sets

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- In other words assign a color to every vertex such that for any edge $(u, v) \in E(G)$, u and v get different colors.

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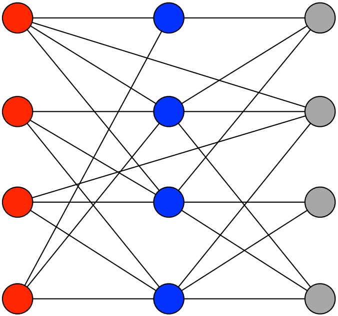

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- An $\mathcal{O}(2^n)$ time algorithm is optimal under some widely believed complexity assumption.

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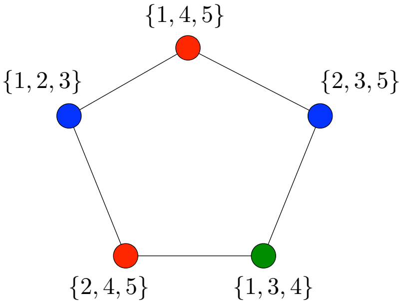

$(3, 1)$-Coloring Example

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- $k = 1, i = 0$, well studied GRAPH COLORING problem.

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- Adjacent vertices share exactly i colors was also studied [BD82].

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- Polynomial time algorithms are known only for cycles, cactus [BDKV14] and bipartite graphs [DZ06].

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  - Parameterized by Treewidth.

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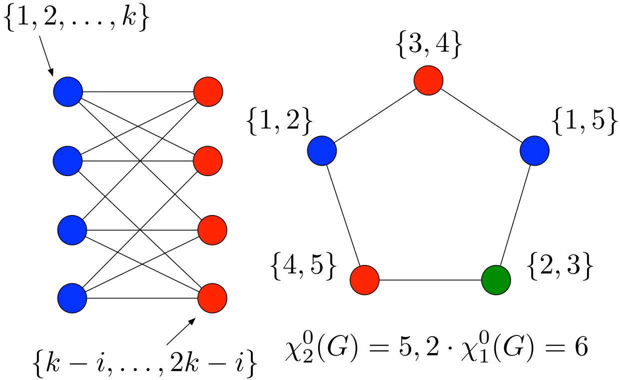

$\chi_2^0(G) = 5, 2 \cdot \chi_1^0(G) = 6$

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- We give exact algorithm to compute $(k, 0)$-Chromatic Number in time $\mathcal{O}^*(2^{kn})$.
- We give exact algorithm to compute $(k, k-1)$-Chromatic Number in time $\mathcal{O}(4^n \cdot n^{\mathcal{O}(1)})$.

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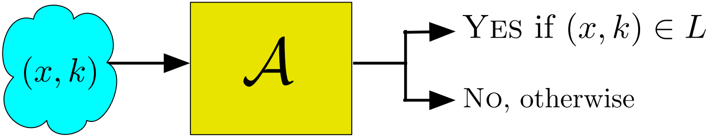
Fixed-Parameter Tractability



- Algorithm $\mathcal{A}$ runs in $f(k) \cdot |x|^c$ time.
- $\mathcal{A}$ is called FIXED PARAMETER ALGORITHM.

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- But, if the parameter is *minimum vertex cover size*, *treewidth*, then it is FPT.

FPT Algorithms

q-(k,i)-COLORING-VERTEX-COVER
Input: An undirected graph G.
Parameter: $vc(G)$ = minimum vertex cover size of G
Question: Does G have a proper (k,i)-coloring with at most q colors?

- A vertex cover Y of size at most k can be computed in $\mathcal{O}(1.27^k \cdot poly(n))$ time.

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- A vertex cover Y of size at most k can be computed in $\mathcal{O}(1.27^k \cdot poly(n))$ time.
- We provide an algorithm that runs in time $\mathcal{O}^*(2^{k|Y|\log(k|Y|)})$ time.

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- For any $v \in I$, there are $\binom{q}{k}$ possible sets that can be assigned.

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- It is known that q-$(1,0)$-coloring is NP-Complete when $q \geqslant 3$.
- But, q-$(1,0)$-coloring is polynomial time solvable on perfect graph.
- Is q-(k,i)-Coloring NP-complete for arbitrary q, k and i?
- Is (k,i)-Coloring polynomial time solvable on clique (more generally on perfect graphs)?

THANK YOU