Paths to Trees and Cacti

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May 24, 2017

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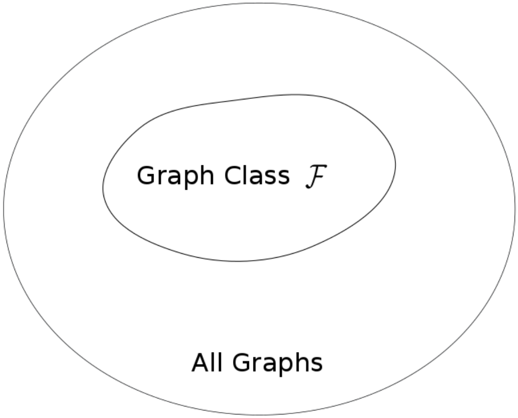
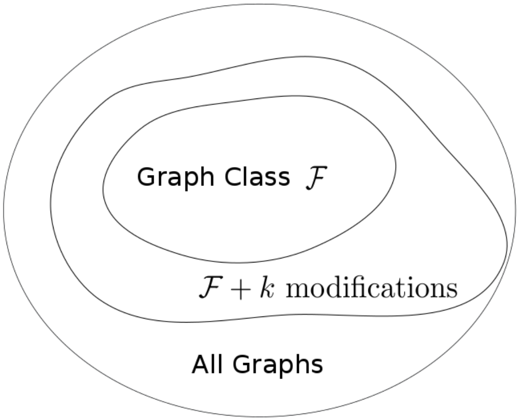

Graph Class $\mathcal{F}$
All Graphs


Graph Class $\mathcal{F}$
$\mathcal{F} + k$ modifications
All Graphs

Graph Modification Problems

$\mathcal{F}$-MODIFICATION
Input: A graph G
Question: Can we obtain a graph in $\mathcal{F}$ by *some* modifications in the graph G?

Modification allowed

- Vertex Deletion
- Edge Deletion
- Edge Addition
- Edge Contraction

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$\mathcal{F}$-Modification : Generalization of many NP-hard problems

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Graph Problem	$\mathcal{F}$	Modification
Vertex Cover	Independent Sets	Vertex Deletion
Feedback vertex set	Forests	Vertex Deletion
Odd cycle transversal	Bipartite Graphs	Vertex Deletion
Minimum Fill-In	Chordal Graphs	Edge Addition
Edge Bipartization	Bipartite Graphs	Edge Deletion
Cluster Editing	Cluster Graphs	Edge Addition & Deletion
Tree Contraction	Trees	Edge Contraction

Outline

Parameterized Complexity & Contraction Problems

TREE CONTRACTION with additional parameter

Contraction as a Partition Problem

Kernel for BOUNDED TREE CONTRACTION

Kernel Lower Bounds

Parameterized Complexity & Contraction Problems

Parameterized Complexity : Quick Overview

- Goal : Find **better** ways to solve NP-hard problems.
- Associate (*small*) parameter k to each instance I.
- Restrict the combinatorial explosion to the parameter k.
- Parameterized problem (I, k) is *fixed-parameter tractable* (FPT) if there is an algorithm that solves it in time $\mathcal{O}(f(k) \cdot |I|^{\mathcal{O}(1)})$.
- Not all problems (for given parameter) admit such an algorithm Hierarchy of classes : FPT $\subseteq$ W[1] $\subseteq$ W[2] $\ldots$

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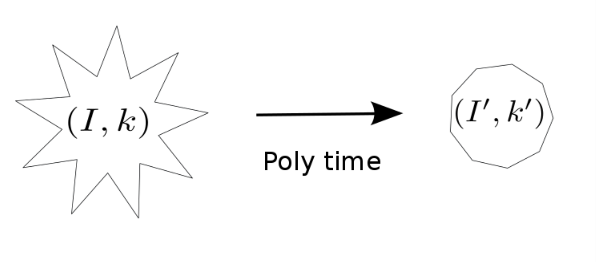
Kernelization : Quick Overview



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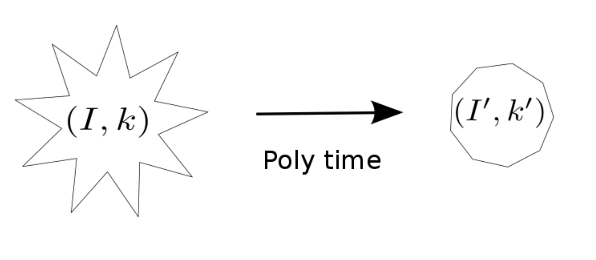

Parameterized problem (I, k) admits a $h(k)$-kernel if there exists a poly-time algorithm $\mathcal{A}$ which given an input (I, k) outputs (I', k') such that

- $|I'| + k' \leq h(k)$
- (I, k) is YES instance iff (I', k') is YES instance

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- *Compression* : If (I', k') is an instance of different problem.

FPT and Kernelization

Problem	FPT	Kernel
VERTEX COVER	$\mathcal{O}(1.27^k \cdot n^2)$	$2k^2$
FEEDBACK VERTEX SET	$\mathcal{O}(3.6181^k \cdot n^c)$	$4k^2$
INDEPENDENT SET	No $f(k) \cdot \lvert I\rvert^{\mathcal{O}(1)}$	No $h(k)$
COLORING	No $f(k) \cdot \lvert I\rvert^{\mathcal{O}(1)}$	No $h(k)$

$\mathcal{F}$-Contraction

$\mathcal{F}$-CONTRACTION **Parameter:** k

Input: A graph G and integer k

Question: Does there exist $F \subseteq E(G)$ of size at most k such that G/F is in $\mathcal{F}$?

$\mathcal{F}$-Contraction : Parameterized Complexity

[Heggernes et al., 2012]	TREE CONTRACTION PATH CONTRACTION	4^k $2^{k+o(k)}$

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[Golovach et al., 2013]	Planar Contraction	FPT
[Cai and Guo, 2013]	Clique Contraction	$2^{\mathcal{O}(k \log k)}$
[Heggernes et al., 2013] [Guillemot and Marx, 2013]	Bipartite Contraction	FPT $2^{\mathcal{O}(k^2)}$

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- *[Agrawal et al., 2017]* $\mathcal{F}$ *is Split Graphs*

Starting Point

Theorem

[Heggernes et al., 2012] TREE CONTRACTION *does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly and* PATH CONTRACTION *admits a linear vertex kernel.*

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Why is there a polynomial kernel for PATHS *but not for* TREES*?*

Tree Contraction with additional parameter

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Bounded Tree Contraction **Parameter:** $k + \ell$

Input: A graph G and integers k, ℓ

Question: Does there exist $F \subseteq E(G)$ of size at most k such that G/F is a tree with at most ℓ leaves?

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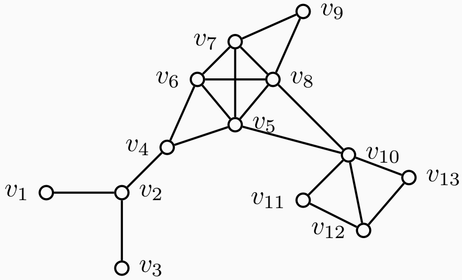
- A kernel with $\mathcal{O}(k\ell)$ vertices and $\mathcal{O}(k^2 + k\ell)$ edges.
- It does not admit better kernel unless NP $\subseteq$ coNP/poly.

Example : $k = 5, \ell = 4$



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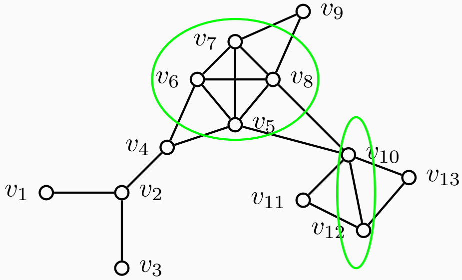

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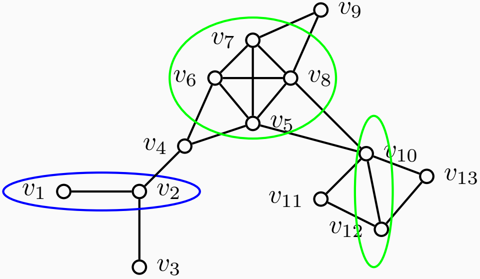

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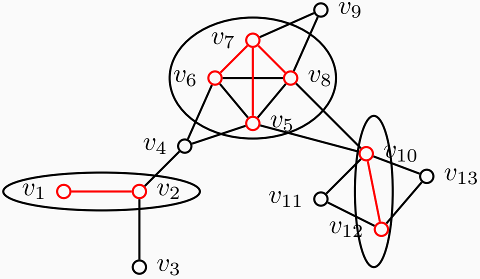
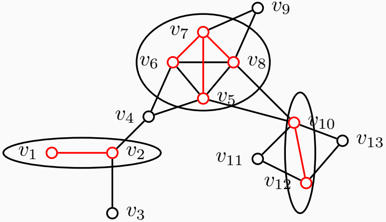
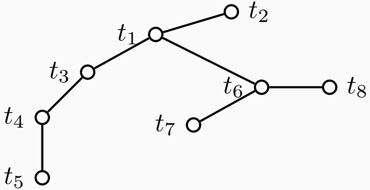




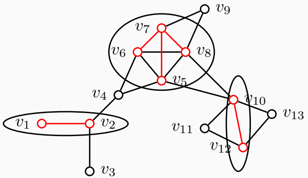
v_9
v_7
v_6
v_8
v_5
v_4
v_{10}
v_{13}
v_1
v_2
v_{11}
v_{12}
v_3

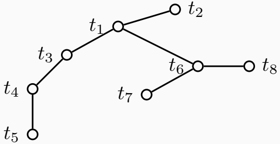


t_2
t_1
t_3
t_6
t_8
t_4
t_7
t_5

Contraction as a Partition Problem

$\mathcal{F}$-Contraction as a Partition Problem

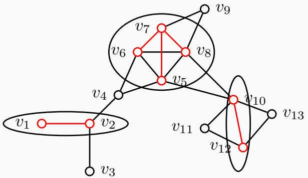


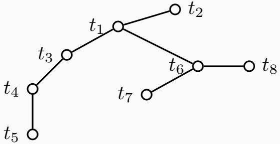


G is contractible to T if there exists a partition of $V(G)$ into $W(t_1), W(t_2), \ldots W(t_{|V(T)|})$ s.t.

- $\forall\ t \in V(T)$, $G[W(t)]$ is connected
- $t_i t_j \in E(T)$ iff $W(t_i)$ and $W(t_j)$ are adjacent in G

$\mathcal{F}$-Contraction as a Partition Problem





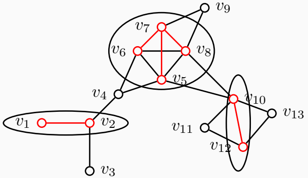
$W(t_1) = \{v_5, v_6, v_7, v_8\}$ $\quad$ $W(t_2) = \{v_9\}$

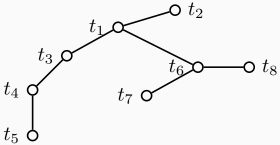
$W(t_3) = \{v_4\}$ $\quad$ $W(t_4) = \{v_1, v_2\}$

$W(t_5) = \{v_3\}$ $\quad$ $W(t_6) = \{v_{10}, v_{12}\}$

$W(t_7) = \{v_{11}\}$ $\quad$ $W(t_8) = \{v_{13}\}$

Witness Structure : Definition





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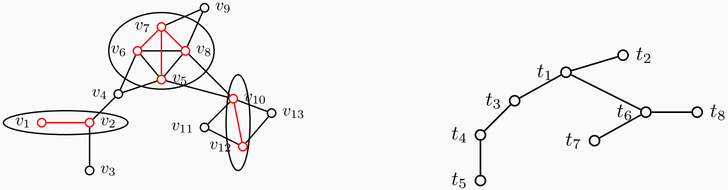

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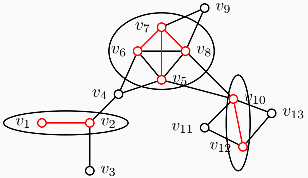

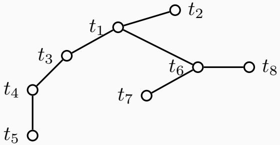

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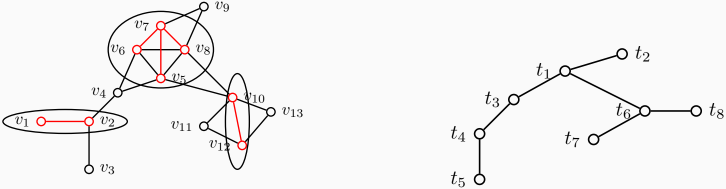
- $k = \sum_{t \in V(T)}(|W(t)| - 1)$
  We say G is *k-contractable* to graph T

Witness Structure : Observations



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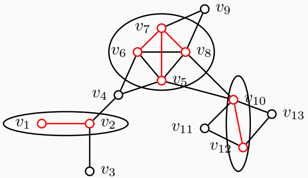

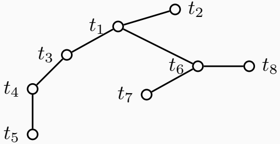

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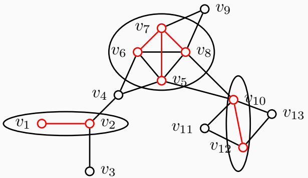

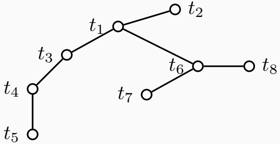

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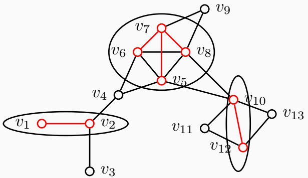

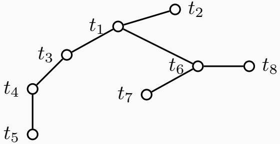

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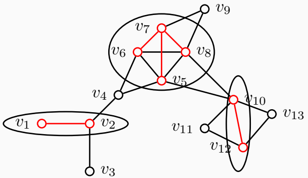

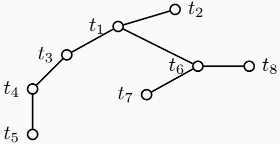

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- $\mathcal{W}$ has at most k big witness sets;
- Union of big witness sets in $\mathcal{W}$ contains at most $2k$ vertices.

Properties of $\mathcal{F}$

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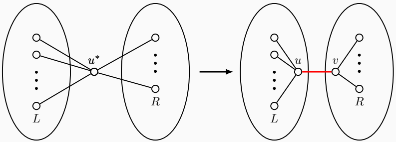
Let $L \cup R$ be a parition of $N(u^*)$.

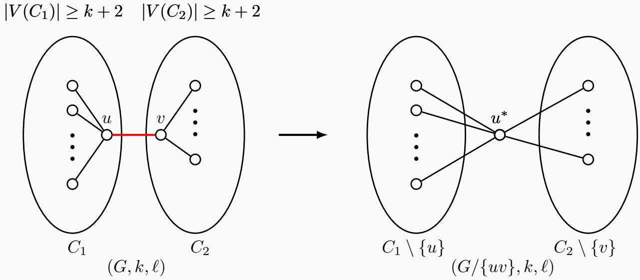
Delete u^* and edge uv s.t. $N(u) = L$ and $N(v) = R$.

Resulting graph is in $\mathcal{F}$



Kernel for Bounded Tree Contraction

Reduction Rule



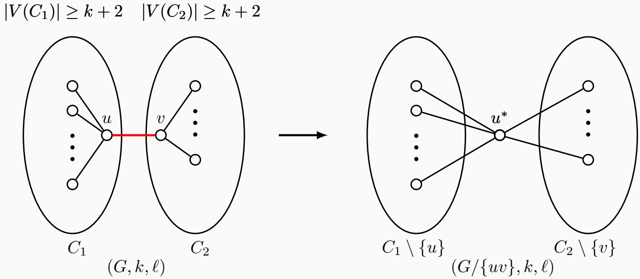
Reduction Rule

Let C_1, C_2 be the connected components in $G - \{uv\}$.

If $|V(C_1)|, |V(C_2)| \geq k+2$ then contract the edge uv.

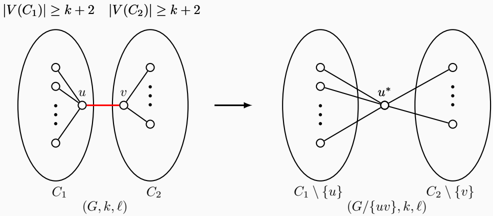
The resulting instance is $(G/\{uv\}, k, \ell)$.

Reduction Rule is sound



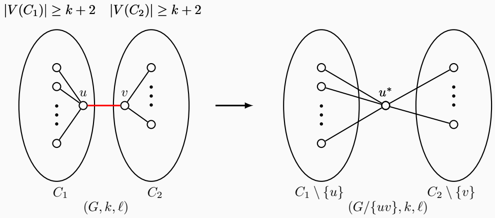
By **Prop 1**, $\mathcal{F}$ is closed under edge contraction

Reduction Rule is complete



- To Prove : uv is not in any minimal solution of size at most k.
- Assume F is a minimal solution of size at most k and $uv \in F$.
- C_1, C_2 are too big to be contained in a witness set.
- t_{uv} is not a leaf in G/F where $u, v \in W(t_{uv})$.
- By **Prop 2**, *uncontract* node t_{uv} and resulting graph is in $\mathcal{F}$.
- This contradicts the minimality of F.

Reduction Rule is complete



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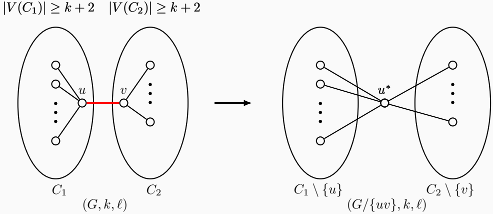

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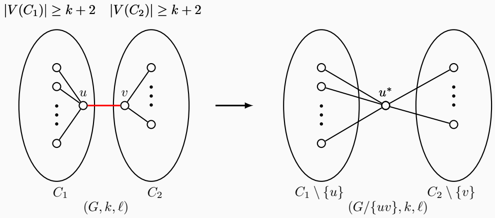

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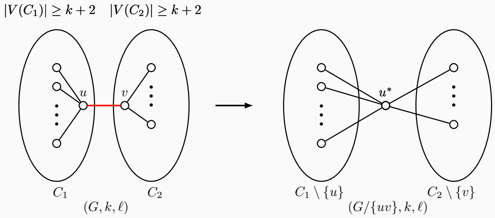

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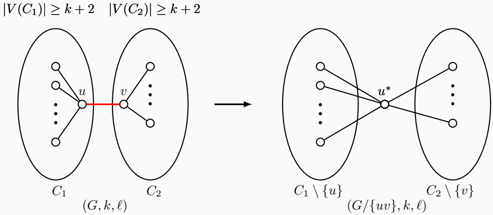

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Kernel for Bounded TC : Bounding $|V(G)|$

- Apply Reduction Rule Exhaustively.
- We bound $V(T)$ and then apply $|V(G)| \leq |V(T)| + k$.
- $|V(T)| \leq \#\text{leaves} \times$ maximum dist. between root and a leaf.
- $P = \{t_1, t_2, \ldots, t_q\}$ be the longest path from root to a leaf.
- If $q \leq 2k + 5$ then $|V(T)| \leq \mathcal{O}(k\ell)$.

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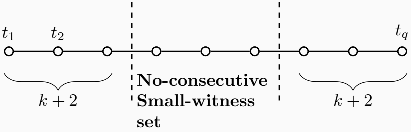

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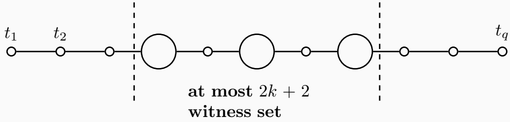

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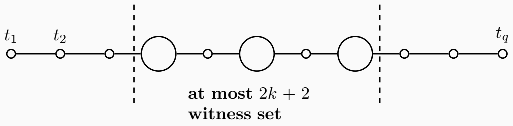

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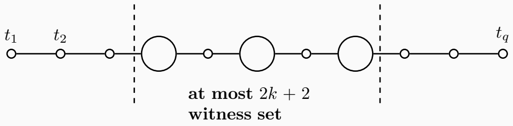

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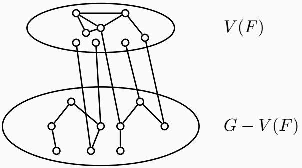

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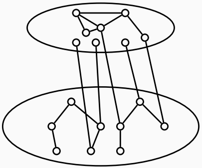

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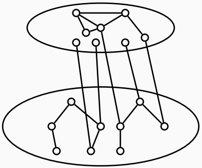

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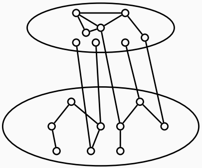

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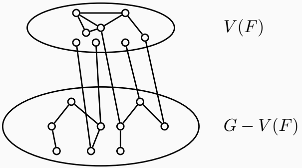

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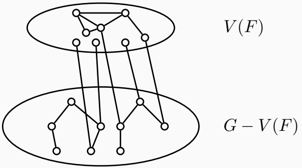

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Theorem

BOUNDED TREE CONTRACTION *admits a kernel of size* $\mathcal{O}(k^2 + k\ell)$.

Extending to other problem

Definition

An *out-tree* T is a digraph where each vertex has in-degree at most 1 and underlying undirected graph is a tree.

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Bounded Out-Tree Contraction *admits a kernel of size* $\mathcal{O}(k^2 + k\ell)$.

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Bounded Cactus Contraction *admits a kernel of size* $\mathcal{O}(k^2 + k\ell)$.

Kernel Lower Bounds

DOMINATING SET **Parameter:** k

Input: Graph G and integer k

Question: Does there exists $X \subseteq V(G)$ of size at most k, such that for each $v \in V(G)$, $X \cap N[v] \neq \emptyset$?

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[Jansen and Pieterse, 2015] proved that DOMINATING SET does not admit a compression of bit size $\mathcal{O}(n^{2-\epsilon})$, for any $\epsilon > 0$ unless NP $\subseteq$ coNP/poly.

Kernel Lower Bounds

Red-Blue Dominating Set **Parameter:** k

Input: Bipartite Graph $G := (R \cup B; E)$ and integer k

Question: Does there exists $X \subseteq R$ of size at most k, such that for each $v \in B$, $X \cap N[v] \neq \emptyset$?

Kernel Lower Bounds

RED-BLUE DOMINATING SET	**Parameter:** k

Input: Bipartite Graph $G := (R \cup B; E)$ and integer k
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By [Jansen and Pieterse, 2015]; RED-BLUE DOMINATING SET does not admit a polynomial compression of bit size $\mathcal{O}(n^{2-\epsilon})$, for any $\epsilon > 0$ unless NP $\subseteq$ coNP/poly.

Kernel Lower Bounds

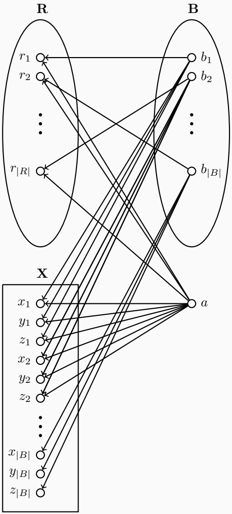

Figure 1: From RBDS to BOUNDED TC

Kernel Lower Bounds

Theorem

BOUNDED TC *does not admit a compression of size* $\mathcal{O}((k^2 + k\ell)^{1-\epsilon})$*, for any* $\epsilon > 0$.

Theorem

BOUNDED CC *does not admit a compression of size* $\mathcal{O}((k^2 + k\ell)^{1-\epsilon})$*, for any* $\epsilon > 0$.

Theorem

BOUNDED OTC *does not admit a compression of size* $\mathcal{O}((k^2 + k\ell)^{1-\epsilon})$*, for any* $\epsilon > 0$.

Thank you!

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