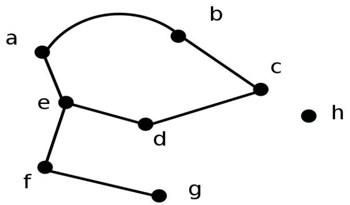
Parameterized and Exact Algorithms for Class Domination Coloring

R. Krithika[1] A. Rai[1] S. Saurabh[1,2] and P. Tale[1]

[1] The Institute of Mathematical Sciences, HBNI, Chennai, India
[2] University Of Bergen, Norway

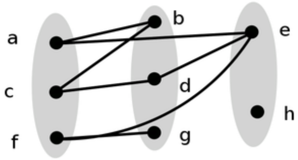
January 16, 2017

Coloring of graph



COLORING
Input: A graph G
Question: Find minimum integer q such that graph G can be partitioned into q independent sets

Coloring of graph



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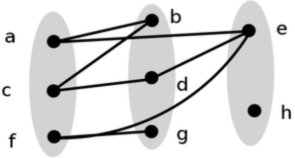
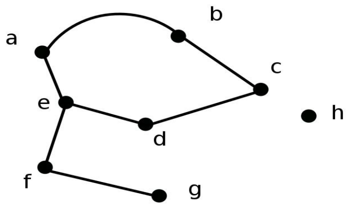


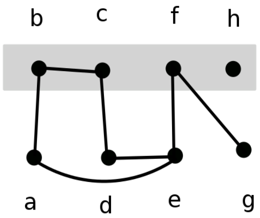
- One of Karp's 21 NP-Complete problems
- Computational Complexity : Determining whether given planer graph (which can be 4-colored) is 3-coloring or not is NP-Complete
- Exact Algorithms : $\mathcal{O}(2^n)$ which optimal under some widely believed complexity assumption
- Parameterized Complexity : Reduction from COLORING to refute existence of certain kind of algorithms

Dominating Set of Graph



DOMINATING SET
Input: A graph G
Question: Find minimum int k such that there exists set dominating set D of cardinality k i.e. $V(G) = N[D]$

Dominating Set of Graph



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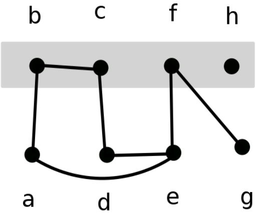
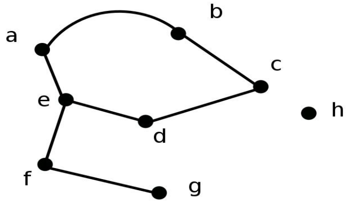


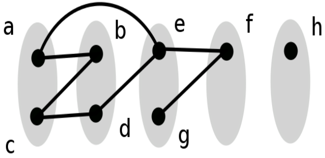
- Exact Algorithm : $\mathcal{O}(1.4969^n)$time and polynomial space.
- $\mathcal{O}(n^k)$ is optimal under certain complexity assumption
- *Complete* for certain classes in parameterized complexity

Class Domination Coloring

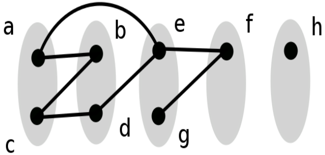


CD-COLORING
Input: A graph G
Question: Find minimum int q such that graph G can be partitioned into q independent sets and every independent set is contained in closed neighbourhood of some vertex

Class Domination Coloring



Class Domination Coloring



- Coloring such that for every color class, there is a vertex that dominates it
- flavour of both Coloring and Dominating Set
- NP-Complete even for Chordal Graphs

Outline

- Parameterized Complexity and Kernalization
  - Short introduction
  - FPT algorithms parameterized by solution size and tree-width
  - FPT algorithms on chordal graphs
  - Kernel for cd-coloring on graphs with girth ≥ 5
- Exact Algorithm
  - $\mathcal{O}^*(2^n)$ algorithm to compute cd-chromatic number
- cd-Partization problem
  - Hardness results
  - On Split graphs
  - Exact algorithm to solve

Parameterized complexity

- Pioneered by Downey and Fellows around 1978
- Goal : Find **better** ways to solve NP-hard problems
- Associate (*small*) parameter k to each instance I
- Restrict the combinatorial explosion to a parameter k
- Parameterized problem (I, k) is *fixed-parameter tractable (FPT)* if there is an algorithm that solves it in time $\mathcal{O}(f(k) \cdot |I|^{\mathcal{O}(1)})$

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CD-COLORING **Parameter:** q
Input: A graph G, integer q
Question: Can graph G be partitioned into q independent sets such that every independent set is contained in closed neighbourhood of some vertex?

Parameterized complexity

Problem	$f(k)$
Vertex Cover(G, k)	$\mathcal{O}(1.27^k \cdot n^2)$
Feedback Vertex Set(G, k)	$\mathcal{O}(3.6181^k \cdot n^c)$
Independent Set(G, k)	No $f(k) \cdot \lvert I \rvert^{\mathcal{O}(1)}$ algorithm
Coloring(G, k)	No $f(k) \cdot \lvert I \rvert^{\mathcal{O}(1)}$ algorithm

Kernelization

- Mathematical analysis of pre-processing
- Goal: Reduce the size of input instance without changing the answer (in polynomial time)
- Parameterized problem (I, k) admits a $h(k)$-kernel if there is a polynomial time algorithm that reduces (I, k) to an equisatisfiable instance (I', k') such that $|I'| + k' \leq h(k)$.

Problem	$h(k)$
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Theorem

A parameterized problem (I, k) is FPT if and only if it admits a kernel.

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- Parameter : Solution size
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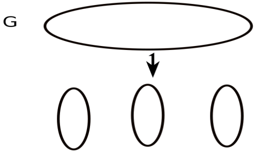
COLORING$(G, 3)$ is NP-complete.



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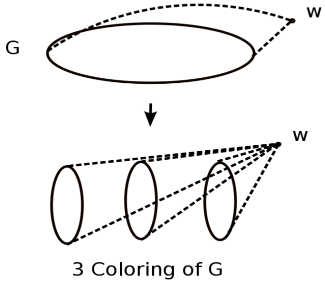

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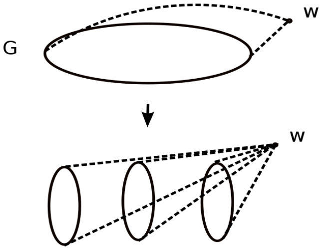
CD-COLORING$(G, 4)$ is NP-complete.



3 Coloring of G

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  - $\Rightarrow \mathcal{O}(f(4) \cdot n^{\mathcal{O}(1)})$ algorithm for CD-COLORING when $q = 4$
  - $\Rightarrow \mathcal{O}(n^{\mathcal{O}(1)})$ algorithm for NP-Complete problem
- para-NP-hard problems

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FPT algorithm on general graph

- Parameter : Treewidth
- Used by Robertson and Seymour in their work on Graph Minors
- Important in algorithm design
- Structural Parameter: measures resemblance with tree
  - #vertices needs to be deleted to get a tree
  - #edges needs to be deleted to get a tree
  - #cycles
  - Separators can be arranged in tree-like fashion
- Treewidth : Size of maximum separator is *best* tree-like arrangement of separator
- Most of NP-hard problems on general graph are polynomial time solvable on trees
- Dynamic Programming works on trees
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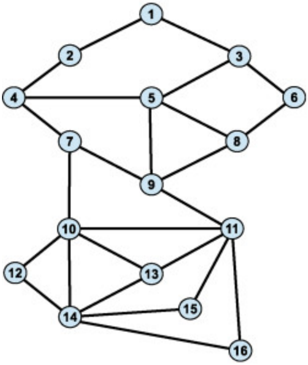
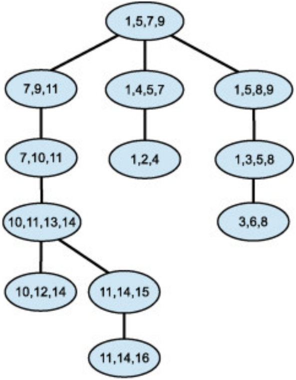
Treewidth



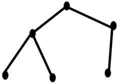
(a) A graph G.



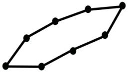
(b) A tree-decomposition of G.

Diagram from 'Metric tree-like structures in real-life networks: an empirical study' by M. Abu-Ata and F. Dragan

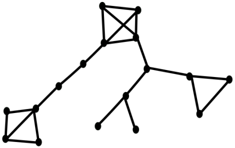
Treewidth



Trees

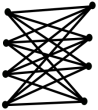


Cycles

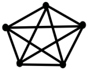


Chordal Graphs

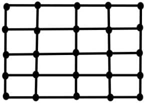
Tree like



Complete Bipartite graphs



Cliques



Grids

Not tree-like

FPT algorithms(treewidth + #colors)

Theorem (Courcelle's theorem, [Cou92])

If ϕ : a graph property that is expressible in $\mathbf{MSO}_2$ *then $\exists$ an algorithm that verifies whether ϕ is satisfied in G in $f(||\phi||, tw(G)) \cdot n$ time.*

- $\mathbf{MSO}_2$: variables for single vertices; single edges; subset of vertices; subset of edges
- f is some computable function
- $\phi(G, q)$: $\mathbf{MSO}_2$ formula which states that G has cd-chromatic number at most q.
- ($\Rightarrow$) CD-COLORING is FPT when parameterized by length of $\phi(G, q)$ plus treewidth of G

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If ϕ : a graph property that is expressible in $\mathbf{MSO}_2$ then $\exists$ an algorithm that verifies whether ϕ is satisfied in G in $f(||\phi||, tw(G)) \cdot n$ time.

- $\mathbf{MSO}_2$: variables for single vertices; single edges; subset of vertices; subset of edges
- f is some computable function
- $\phi(G, q)$: $\mathbf{MSO}_2$ formula which states that G has cd-chromatic number at most q.
- ($\Rightarrow$) CD-COLORING is FPT when parameterized by length of $\phi(G, q)$ plus treewidth of G

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What is the class of graphs on which it is FPT when parameterized by #colors (only)?

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- Let $\omega(G)$ be the size of a maximum clique in G
- For a graph G, $tw(G) \geq \omega(G) - 1$
- For a chordal graph G, $tw(G) = \omega(G) - 1$
- Finding $w(G)$ is poly-time in chordal graphs
- No two vertices in a clique can be in the same color class of cd-coloring.
  - if $\omega(G) \geq q$ then (G, q) is NO instance of CD-COLORING
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- If $G_1, \ldots, G_l$ are the connected components of G, then $\chi_{cd}(G) = \sum_{i=1}^{l} \chi_{cd}(G_i)$.
- wlog assume that input graph is connected
- Let $\chi_{cd}(G)$ be the minimum number of colors in any cd-coloring of graph G
- Dominating Set : $S \subseteq V(G)$ s.t. $V(G) = \bigcup_{v \in S} N[v]$
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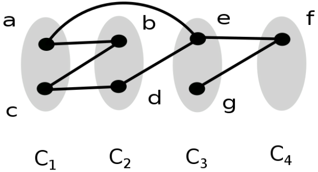
FPT algorithms and Kernalization



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which dominates color classes C_1, C_2, C_3, C_4 resp.

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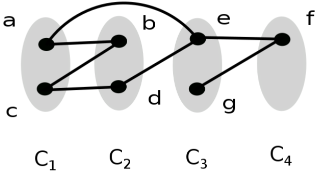

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FPT algorithm for $\mathcal{G}_5$

- $\mathcal{G}_5 = \{G \mid \text{girth of } G \geq 5\}$
- For a graph $G \in \mathcal{G}_5$, for any u, v: $|N(v) \cap N(u)| \leq 1$.
- In any cd-coloring of G, every color class has unique vertex which dominates it
- For general graphs : $minTDS(G) \leq \chi_{cd}(G)$
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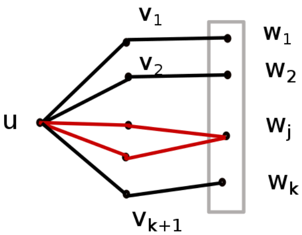
Kernalization for Total-Dom-Set

Claim

For $G \in \mathcal{G}_5$, if $deg(u) \geq k+1$, then any total dominating set of size at most k contains u.

- Consider Total Dominating Set $\{w_1, w_2, \ldots, w_k\}$ which doesn't contain u
- Note : v_i may be equal to w_j
- By Pigeon-hole principle, there exists some w_j which is adjacent to two vertices in $N(u)$
- This contradicts the fact that $G \in \mathcal{G}_5$



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Partition graph into 3 parts: High degree vertices which will be part of any solution (H), vertices which have been dominated by partial solution (J) and rest of the graph(R)

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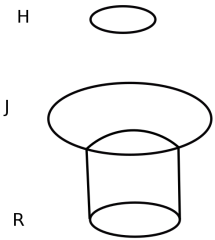
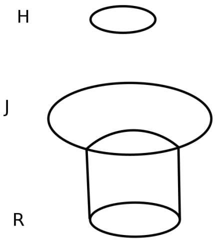


- $H = \{u | \ deg(u) \geq k + 1\}$
  $J = N[H] \setminus H$
  $R = V(G) \setminus (H \cup J)$

Kernalization for Total-Dom-Set

Partition graph into 3 parts: High degree vertices which will be part of any solution (H), vertices which have been dominated by partial solution (J) and rest of the graph(R)



- $H = \{u | \; deg(u) \geq k+1\}$
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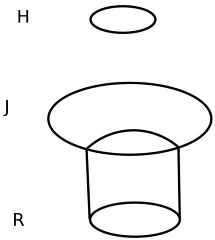

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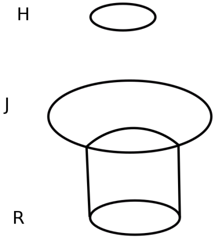

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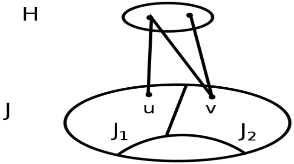
- $|J \cup R| \leq |R| * k \leq \mathcal{O}(k^3)$

Reduction Rule

For $u, v \in J \setminus N(R)$, if $N(u) \cap H \subseteq N(v) \cap H$ then delete u.



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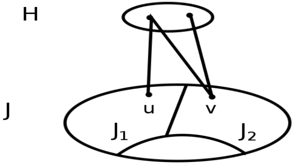

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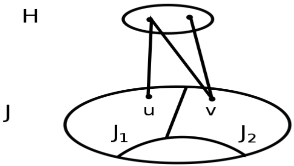

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Partition $J \setminus N(R)$ into J_1 and J_2

$J_1 = \{u |$ s.t. $|N(u) \cap H| = 1\}$

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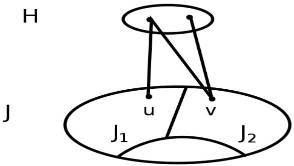

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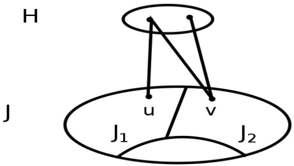

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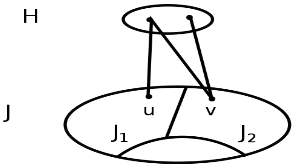

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Lemma

Total Dominating Set *admits a kernel of $\mathcal{O}(k^3)$ vertices on $\mathcal{G}_5$.*

Exact algorithm to compute cd-chromatic number

Let $\mathcal{I}$ be the set containing all independent sets in G

$$\mathcal{I}^1 = \{X \mid X \in \mathcal{I} \text{ and } \exists u \in V(G) \text{ s.t. } X \subseteq N(u)\}$$

$\mathcal{I}^1$: Possible candidates for color classes in cd-coloring of graph

$$\mathcal{I}^2 = \{Y \mid \exists X_1, X_2 \in \mathcal{I}^1 \text{ s.t. } X_1 \cup X_2 = Y \text{ and } X_1 \cap X_2 = \emptyset\}$$

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Set and its representation

Universe $U = \{u_1, u_2, \ldots, u_n\}$ with fixed ordering on elements.

Characteristic vector of a set $S \subseteq U$ is bit vector $\psi(S)$, of length n s.t. $\psi(S)[j] = 1$ iff $u_j \in S$

Hamming weight of a vector ϕ is the number of 1s in ϕ and its denoted by $\mathcal{H}(\phi)$

val(ϕ) denotes the integer d of which ϕ is the binary representation.

Let z be an indeterminate variable. For $S_1, S_2 \subseteq U$, define modefied multiplication $(\star)$

$$z^{val(\psi(S_1))} \star z^{val(\psi(S_2))} = \begin{cases} z^{val(\psi(S_1))+val(\psi(S_2))} & \text{if } S_1 \cap S_2 = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Objective : To compute $z^{val(\psi(S_1))} \star z^{val(\psi(S_2))}$ without explicitely looking at S_1, S_2

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$S_3 = \{c, d\}$	$B_3 = 0011$	2
$S_1 \cup S_2 = \{a, b, c\}$	$B_{12} = 1110$	3
$S_1 \cup S_3 = \{a, c, d\}$	$B_{13} = 1011$	3

$$S_1 \cap S_2 = \emptyset \text{ and } \mathcal{H}(B_{12}) = \mathcal{H}(B_1) + \mathcal{H}(B_2)$$
$$S_1 \cap S_3 \neq \emptyset \text{ and } \mathcal{H}(B_{13}) \neq \mathcal{H}(B_1) + \mathcal{H}(B_2)$$

1-bit is *lost* while adding two bit-vectors B and B' if there is an index $i \in [n]$ such that $B[i] = B'[i] = 1$ i.e. while adding two bit-vectors corresponds to sets which are not disjoint

Algorithm to compute $(\star)$

Algorithm to compute modified multiplication $(\star)$

Input: Two polynomials $p(z), r(z)$
Output: $p(z) \star r(z)$

Initialize polynomial $t(z)$ to 0
for *each ordered pair* (i, j) *such that* $i + j \leq n$ **do**
    $s_i(z) \leftarrow$ monomials in $p(z)$ whose exponent as Ham-Wt i
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    Compute $s_{ij}(z) = s_i(z) * s_j(z)$ (standard multiplication)
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Exact Algorithm

Maximum degree of $q(z)$ is 2^n ($\Rightarrow$) running time $\mathcal{O}(n^2 \times 4^n)$

Lemma (Fast Fourier Transform [SS71])

Two polynomials of degree at most d over any commutative ring $\mathcal{R}$ can be multiplied using $\mathcal{O}(d \cdot \log d \cdot \log\log d)$ additions and multiplications in $\mathcal{R}$.

($\Rightarrow$) running time $\mathcal{O}(n^2 \times 2^n \cdot n \cdot \log n)$
Since $\chi_{cd}(G) \leq |V(G)|$, we need to multiply two polynomials at most n times

Theorem

Given a graph G on n vertices, there is an algorithm which finds its cd-chromatic number in $\mathcal{O}(2^n n^4 \log n)$ time.

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cd-Partization

> CD-PARTIZATION **Parameter:** k, q
> **Input:** Graph G, integers k and q
> **Question:** Does there exist $S \subseteq V(G)$, $|S| \leq k$, such that $\chi_{cd}(G - S) \leq q$?

Theorem

q-CD-PARTIZATION is NP-complete for $q \in \{2, 3\}$.

- Note $\mathcal{G} = \{G \mid \chi_{cd}(G) \leq q\}$ is not a hereditary graph class
- Result of Lewis and Yannakakis [LY80] does not imply NP-hardness.
- Reduction from q-PARTIZATION

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Split Graph : Vertex set can be partitioned into a clique and an independent set.

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CD-PARTIZATION *on split graphs is* FPT *with respect to parameters q and k. Furthermore, the problem does not admit a polynomial kernel unless* NP $\subseteq$ coNP/poly*.*

Exact Algorithms for cd-Partization

Theorem

Given a graph G and an integer k, there is an algorithm that determines if there is a set S of size k whose deletion results in a graph H with $\chi_{cd}(H) \leq 3$ in $\mathcal{O}^*(2.3146^k)$ time.

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Given a graph G and an integer k, there is an algorithm that determines if there is a set S of size k whose deletion results in a graph H with $\chi_{cd}(H) \leq 3$ in $\mathcal{O}^*(2.3146^k)$ time.

- Complete characterization of class $\mathcal{H} = \{H \mid \chi_{cd}(H) \leq 3\}$
- Use exact algorithms for Vertex Cover and Odd Cycle Transversal as sub-routine

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Doz Dr A Schönhage and Volker Strassen.
Schnelle Multiplikation Grosser Zahlen.

Thank you!