Subset Feedback Vertex Set in Chordal and Split Graphs

Geevarghese Philip [1], Varun Rajan [1], Saket Saurabh [2,3], and Prafullkumar Tale [3]

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[1] Chennai Mathematical Institute, Chennai, India,
[2] University of Bergen, Bergen, Norway
[3] The Institute of Mathematical Sciences, HBNI, Chennai, India

The Problem

FEEDBACK VERTEX SET(FVS) **Parameter:** k
Input: A graph $G = (V, E)$, and an integer k
Question: Does there exist a set $S \subseteq V$ of at most k vertices of G such that the subgraph $G[V \setminus S]$ contains no cycle?

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T*-cycle* is a cycle which contains at least one vertex from T.

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2016 Hols and Kratsch [5] obtained a randomized polynomial kernel with $\mathcal{O}(k^9)$ vertices.

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The problem is NP-Complete even on split graphs [4].

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SUBSET-FVS IN CHORDAL to 3-HITTING SET.

(*Parameter Preserving Reduction*)

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∘ 3-Hitting Set is FPT (running time $2.076^k \cdot n^{\mathcal{O}(1)}$ [8])

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Our Results

Theorem

SUBSET-FVS IN SPLIT *admits a kernel with* $\mathcal{O}(k^2)$ *vertices and* $\mathcal{O}(k^2)$ *edges.*

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Kernel : Overview

Overview of Kernelization

(K, I) is a split partition ($K-$ clique and $I-$ independent set)

Step 1: Reduce the input to an instance $(G; T; k)$ where the terminal set T is exactly the independent set I

Step 2: If $v \in K$ has at least $k+1$ neighbours in I then either include v in a solution or delete an edge incident with v;
Each $v \in K$ has at most k neighbours in I.

Step 3: Bound the number of vertices in K by $10k$;
An instance with $\mathcal{O}(k^2)$ vertices in I.

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Kernel : Step 1

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- If there is a terminal vertex t on the clique side and clique side is $\geq k+3$ then include t in solution.

After Simplification

1. Each vertex in G has degree at least two.
2. Every vertex in G is part of some T-triangle.
3. Let (K, I) be the split partition of G. Then $T = I$ and every vertex in K has a neighbour in I.

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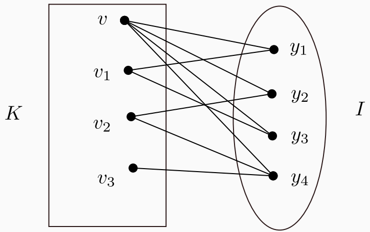
It is enough to bound

- the number of adjacent vertices in independent set for each vertex in clique-side, and
- the size of clique-side.

Kernel : Step 2 (Reducing neighbours in Ind-Set side)

For a vertex $v \in K$ on the clique side define

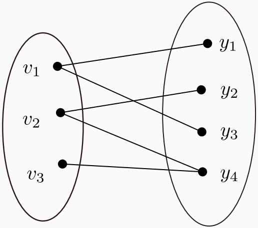
- First Nbrs: the set of neighbours on the independent side I.
  $N_1(v) = N(v) \cap I$
- Second Nbrs: the second neighbourhood "going via I".
  $N_2(v) = N(N_1(v)) \setminus \{v\}$.



$N_1(v) = \{y_1, y_2, y_3, y_4\}$

$N_2(v) = \{v_1, v_2, v_3\}$

Bipartite graph $(B(v))$ *corresponding to* vertex $v \in K$: Graph obtained from $G[N_1(v) \cup N_2(v)]$ by deleting every edge with both its endvertices in $N_2(v)$.



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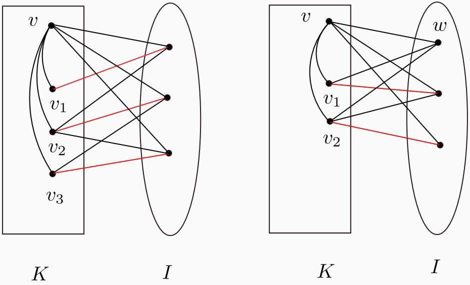

- Too many vertex disjoint triangles intersecting at v.
- Matching edges are enough to store information regarding T-cycles using vw.

Maximum Matching is $\geq k + 1$

Reduction Rule

If there is a vertex v on the clique side K of graph G s.t. the bipartite graph $B(v)$ has a matching of size at least $k + 1$ then include vertex v in solution.

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We need Expansion Lemma to handle second case.

Expansion Lemma and its Matching version

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- every vertex of X is incident with exactly t edges of M, and
- the number of vertices in Y which are incident with at least one edge in M is exactly $t|P|$.

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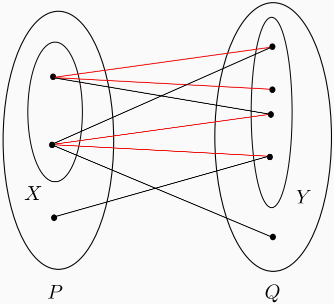

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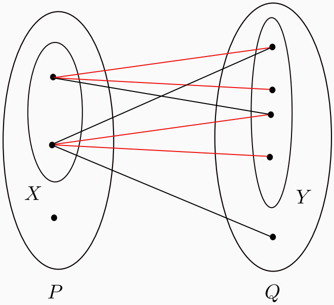
Then there exist nonempty sets $X \subseteq P$ and $Y \subseteq Q$ s.t.

- X has a t-expansion into Y, and
- no vertex in Y has a neighbour outside X.



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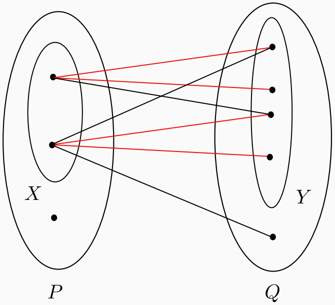

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X and Y can be found in poly-time.

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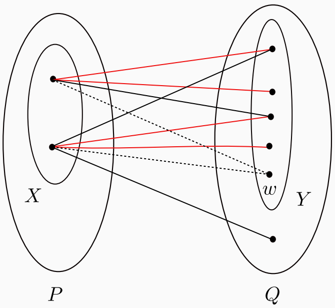

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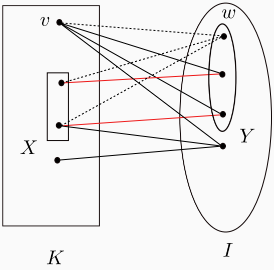
- M does *not* saturate w, and
- $N_G(Y) = X \cup \{v\}$.



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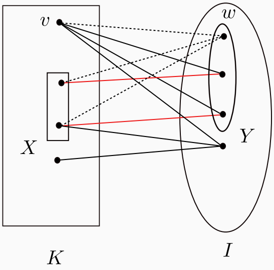

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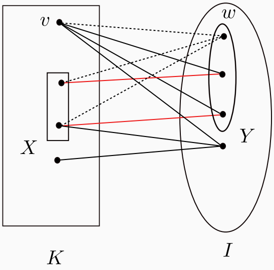

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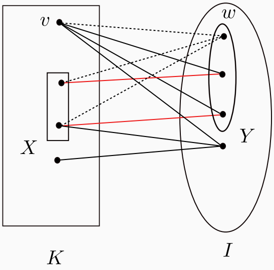
Case (A): Solution picks v.

All T-cycles containing edge vw are killed.



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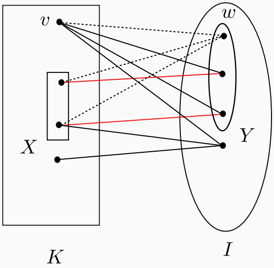

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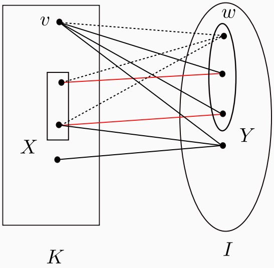

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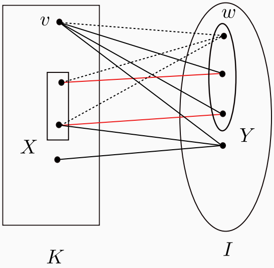

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Modify solution: Remove Y; Add remaining vertices in X.

Kernel : Step 3 (Bounding the Size of the Clique Side)

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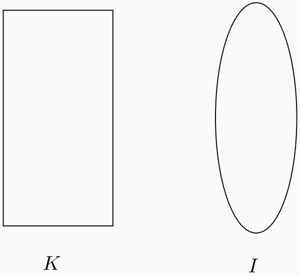

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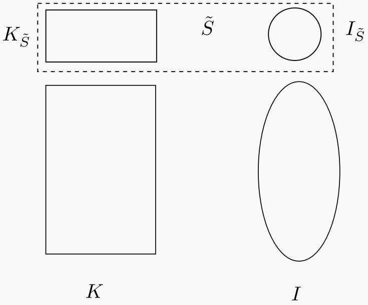

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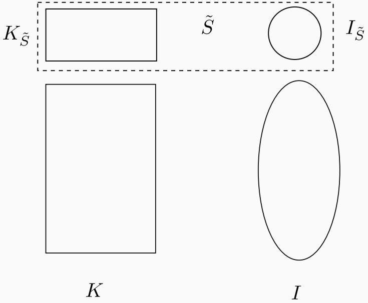

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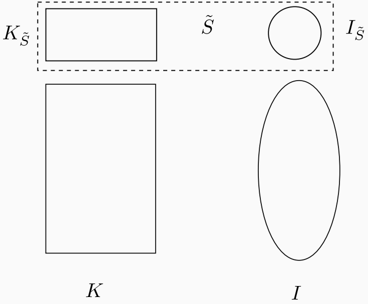
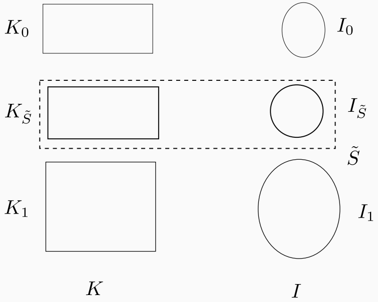
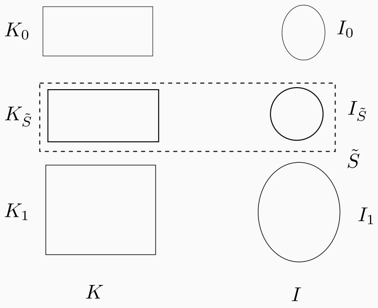
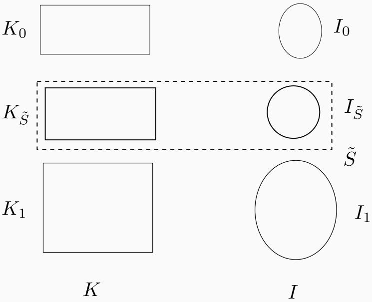

$K_{\tilde{S}}$
$\tilde{S}$
$I_{\tilde{S}}$
K
I

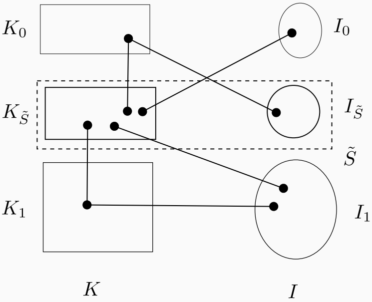


- K_0 : set of clique-side vertices not in $\tilde{S}$ whose neighbourhoods in the independent-side I are all contained in $I_{\tilde{S}}$.
  $K_0 = \{u \in (K \setminus K_{\tilde{S}})\ ;\ N(u) \cap I \subseteq I_{\tilde{S}}\};$
- I_0 : set of independent-side vertices not in $\tilde{S}$ whose neighbourhoods are all contained in $K_{\tilde{S}}$.
  $I_0 = \{v \in I \setminus I_{\tilde{S}}\ ;\ N(v) \subseteq K_{\tilde{S}}\}$

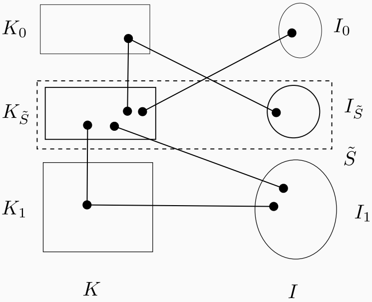

K_0
I_0
$K_{\tilde{S}}$
$I_{\tilde{S}}$
$\tilde{S}$
K_1
I_1
K
I



- K_1 : Remaining vertices in K.
  $K_1 = K \setminus (K_{\tilde{S}} \cup K_0)$.
- I_1 : Remaining vertices in I.
  $I_1 = I \setminus (I_{\tilde{S}} \cup I_0)$.

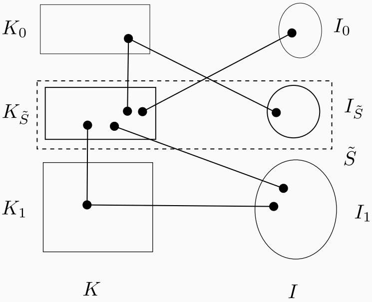


Some observations:



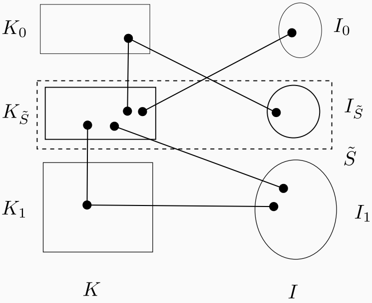
Some observations:

- $|K_{\tilde{S}}| \leq 2k$ and $|I_{\tilde{S}}| \leq k$.



Some observations:

- $|K_{\tilde{S}}| \leq 2k$ and $|I_{\tilde{S}}| \leq k$.
- Each vertex in K_1 has (i) no neighbour in I_0 and (ii) at least one neighbour in I_1.



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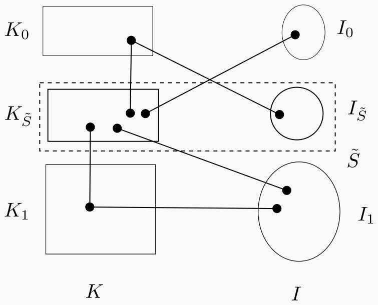
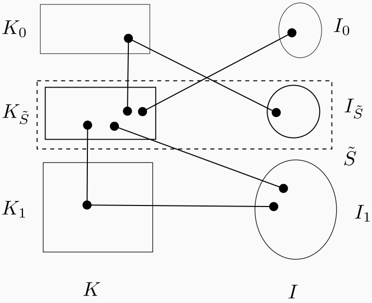


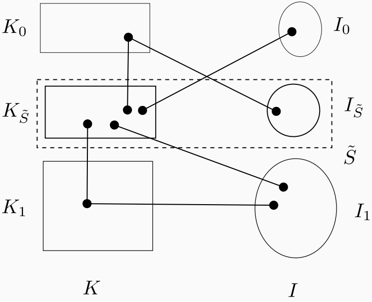
Some observations:

- Each vertex in I_1 has exactly one neighbour in K_1.



Some observations:

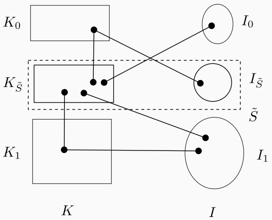
- Each vertex in I_1 has exactly one neighbour in K_1.
- The bipartite graph obtained from $G[K_1 \cup I_1]$ by deleting all the edges in $G[K_1]$ is a forest where each connected component is a star.

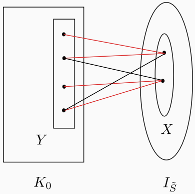


We bound

- K_0 using 2-expansion on graph across K_0 and $I_{\tilde{S}}$.
- K_1 using 2-expansion on graph across K_1 and $\tilde{S}$

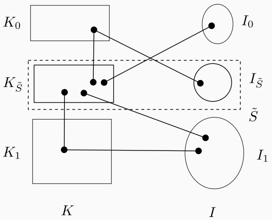
Bounding $|K_0|$

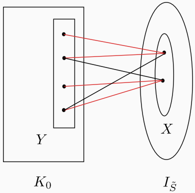




2-expansion across $I_{\tilde{S}}$ and K_0.

Bounding $|K_0|$





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Solution intersects T-triangles in $X \cup Y$.

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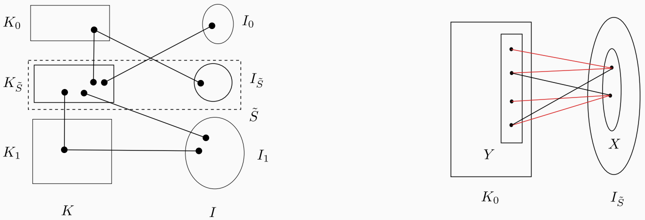

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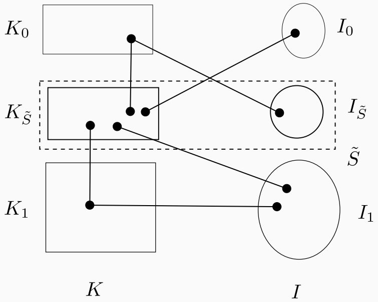
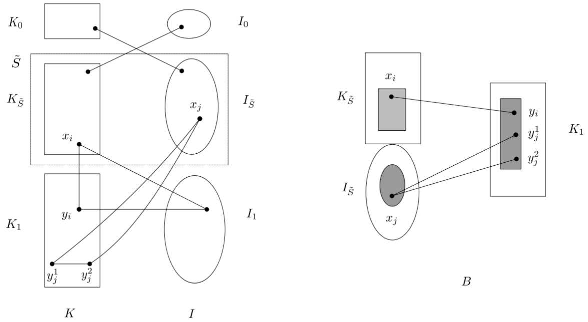
Solution intersects T-triangles in $X \cup Y$.

Since K_0 (and hence Y) interact with only $I_{\tilde{S}}$ (and hence only X), it is safe to pick all vertices in X to kill T-triangles in $X \cup Y$.

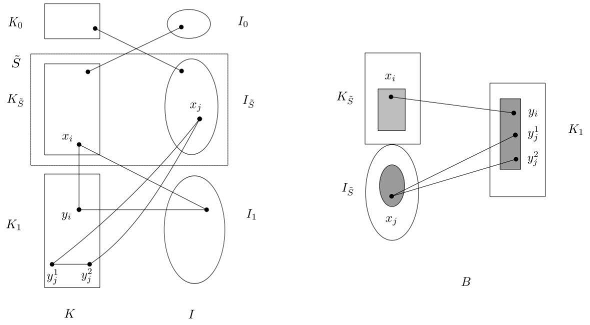
Bounding $|K_1|$



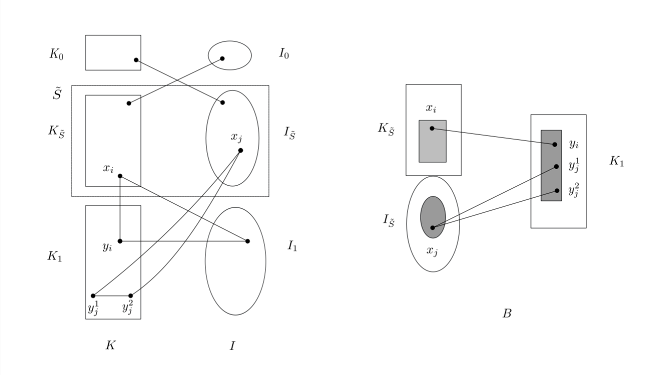
Construct auxilary bipartite graphs B across $\tilde{S}$ and K_1.



K_0
I_0
$\tilde{S}$
$K_{\tilde{S}}$
x_j
$I_{\tilde{S}}$
x_i
I_1
y_i
K_1
y_j^1
y_j^2
K
I
x_i
$K_{\tilde{S}}$
y_i
y_j^1
K_1
y_j^2
$I_{\tilde{S}}$
x_j
B

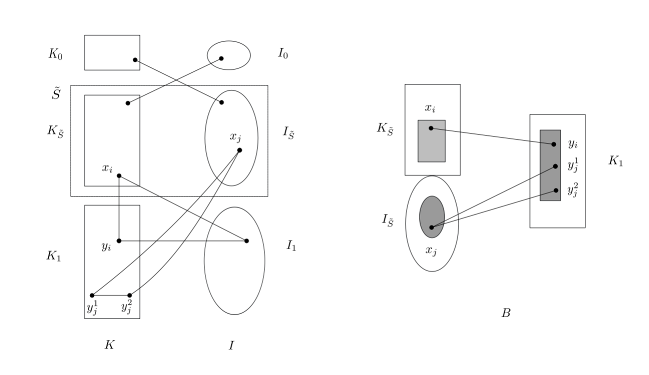


2-Expansion across $\tilde{S}$ and K.

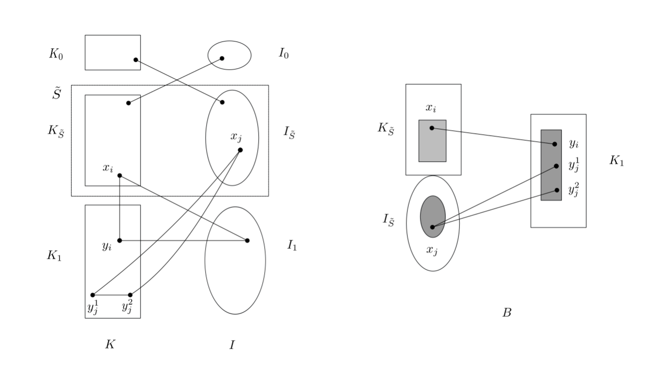


2-Expansion across $\tilde{S}$ and K.

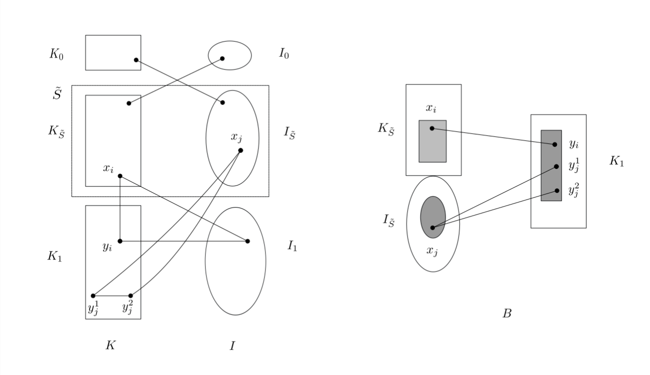
Shaded regions in $\tilde{S}$ and K_1 represent sets X, Y, respectively.



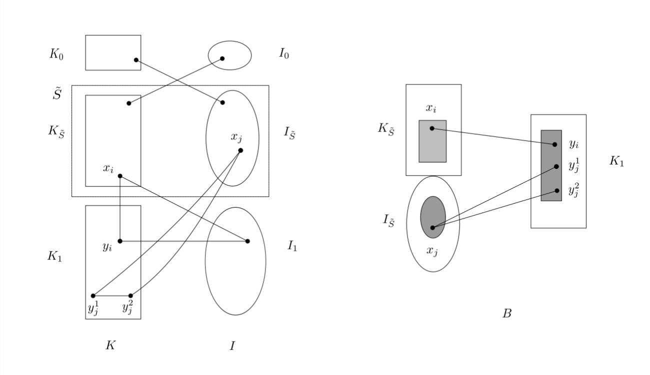
Each vertex $x_i \in (X \cap K_{\tilde{S}})$ is a part of T-triangle with an edge in M.



Each vertex $x_j \in (X \cap I_{\tilde{S}})$ is part of T-triangle with two edges in M.



We get $|X|$ many pairwise vertex-disjoint T-triangles.



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Let S be an optimum solution. We modify this to exclude all vertices in Y.

Theorem

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- We only use expansion lemma which can be applied in poly-time.

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- Interesting to investigate other implicit hitting set problems from graph theory and obtain better kernel and FPT results than the ones guaranteed by HITTING SET.

Thank you!

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