

# Lossy Kernels for Graph Contraction Problems

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# Graph Modification Problems

## $\mathcal{H}$ - Modification Problems

**Input** : Graph  $G$ , integer  $k$

**Para** :  $k$

**Output** : Can we make at most  $k$  modifications in  $G$  so that resulting graph is in  $\mathcal{H}$ ?



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Modification — (1) vertex deletion      (2) edge deletion  
(3) edge addition      (4) edge contraction



$\mathcal{H}$ -Modification Problems  $\equiv$   $\mathcal{F}$ -Free-Modification Problems

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Problem	$\mathcal{H}$	$\mathcal{F}$ -Free	Modification
VERTEX COVER	Empty Graphs	$P_2$	Vertex Deletion
EDGE BIPARTIZATION	Bipartite Graphs	$C_3, C_5, C_6, \dots$	Edge Deletion
MINIMUM FILL-IN	Chordal Graphs	$C_4, C_5, C_6, \dots$	Edge Addition
CLUSTER EDITING	Set of Cliques	$P_3$	Edge Add + Del.



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**Theorem (Cai 1996):** If  $\mathcal{F}$  is bounded and (1), (2), and/or (3) then  
 $\mathcal{F}$ -Free -Modification Problems are FPT.

- VERTEX COVER is FPT
- CLUSTER VERTEX EDITING is FPT.



Modification — (1) vertex deletion      (2) edge deletion  
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If  $\mathcal{F}$  is infinite and (1), (2), and/or (3) then

$\mathcal{F}$ -Free -Modification Problems are FPT.

- MINIMUM FILL-IN is FPT (Cai '96 + Kaplan '96).
- EDGE BIPARTIZATION is FPT (S. Wernicke 2003).



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*$\mathcal{F}$* -Free -Modification Problems when only (4) is allowed?



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**$\mathcal{F}$ -Free -Modification Problems** when only (4) is allowed?

TREE CONTRACTION	$4^k$	No poly kernel	Heggernes et al. (2011)
CLIQUE CONTRACTION	$\exp(k \log(k))$	No poly kernel	Cai et al. (2013)
PLANAR CONTRACTION	FPT	—	Golavach et al. (2013)
BIPARTITE CONTRACTION	$\exp(k^2)$	—	Guillemot + Mark (2013)



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Can we have generic theorem as that of Cai 1996?



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**Theorem** (Lokshtanov et al. 2013 + Cai et al. 2013):

If  $\mathcal{F}$  contains only one graph (a path on  $\geq 5$  or cycle on  $\geq 4$  vertices) then

$\mathcal{F}$ -Free-Modification Problems are  $W[2]$ -hard.



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CHORDAL CONTRACTION is  $W[2]$ -hard.

**Theorem** (Agrawal et al. 2017): SPLIT CONTRACTION is  $W[1]$ -hard.



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No FPT algorithm for CHORDAL CONTRACTION , SPLIT CONTRACTION

No poly kernel for CLIQUE CONTRACTION



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$\alpha > 1$



$\alpha$ -FPT approximation algorithms

$\alpha > 1$

$\alpha$ -lossy kernel



$$\alpha > 1$$

## $\alpha$ -FPT approximation algorithms

- \* Runs in FPT time
- \* If there is a solution  $X$  of size at most  $k$  then returns a solution of size at most  $\alpha|X|$ .

## $\alpha$ -lossy kernel



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## $\alpha$ -lossy kernel

- \* On input  $(G, k)$  produces output  $(G', k')$  in polynomial time.
- \* Given  $c$ -factor solution  $S'$  for  $(G', k')$  produces an  $(\alpha c)$ -factor solution  $S$  for  $(G, k)$  in polynomial time.
- \* Allowed to fail if  $S'$  is really bad.



$\alpha$ -FPT approximation algorithms

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$\alpha > 1$  there is **no**  $\alpha$ -FPT approx algo for **CHORDAL CONTRACTION** unless  $\text{FPT} \neq \text{W}[1]$ .

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$\alpha > 2$  there is an  $\alpha$ -FPT approx algo for **SPLIT CONTRACTION**.

$1.25 > \alpha$  there is **no**  $\alpha$ -FPT approx algo for **SPLIT CONTRACTION** under Gap-ETH.

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**Theorem** : For any  $\alpha > 1$ , **CLIQUE CONTRACTION** parameterised by the size of solution  $k$  admits an  $\alpha$ -lossy kernel with  $O(k^{d+1})$  vertices where  $d = 1/\alpha$ .

$1.25 > \alpha$  there is no  $\alpha$ -FPT approx algo for **SPLIT CONTRACTION** under Gap-ETH.

**Theorem** : Assuming Gap-ETH, no FPT algorithm can approximate **SPLIT CONTRACTION** within a factor of  $\alpha$ , for any  $\alpha < 1.25$ .



# CLIQUE CONTRACTION

(  $\alpha$  -lossy kernel of polynomial size )



# CLIQUE CONTRACTION

**Input** : Graph  $G$ , integer  $k$

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**Output** : Output a set  $F$  of edges of minimum cardinality s.t.  
 $G/F$  is a clique.



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 $G/F$  is a clique.

Every solution of size larger than  $k$  is equally bad.



- \* allowed to fail (see board)
- \* w.l.o.g  $G$  is connected and has at least  $(k + 3)$  vertices
- \* any spanning tree is a trivial solution
- \* Trivial “no” instance : (Path on four vertices, 1)







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- \* 2-factor approx algo for vertex cover in compliment graph



**Reduction Rule 1 :** Given  $(G, k)$  find minimum sized  $X$  such that  $G-X$  is a clique.  
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**Solution lifting algorithm :**

- If un-changed, return same solution.
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- return a large solution (spanning tree)







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For every subset  $X'$  of  $X$ , mark a vertex which is common neighbour.

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Kernel of size  $O(2^k)$  for CLIQUE CONTRACTION.



## Role of vertices in $Y$

$$\frac{d+1}{d} \leq \alpha$$

- Provide connectivity to witness sets coming out of  $X$

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For every subset  $X'$  of  $X$  of size  $\leq d$ , mark a vertex which is common neighbour.

- Dictate witness sets in  $X$

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**Reduction Rule 2:** Mark vertices in  $Y$  and delete unmarked vertices.

RR-2 runs in poly-time on large instances.



**Reduction Rule 2:** Mark vertices in  $Y$  and delete unmarked vertices.

$Y'$  — deleted vertices.  $(G' = G \setminus Y', k)$  is new instance.

$F'$  is given solution for  $(G', k)$

**Solution Lifting Algorithm:** Add a vertex in  $Y$  to witness sets completely in  $X$  which are of size  $\geq d$ .

**Lemma:** Size of obtained solution  $F$  is  $\leq \alpha |F'|$ .



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**Lemma :** Size of obtained solution  $F$  is  $\leq \alpha |F'|$ .

- Such vertex exists because of first marking
- Adding an extra edge for every  $d$  vertices
- Every witness set of size  $\leq d$  in  $X$  is adjacent with everything in  $Y'$  because of second marking



**Lemma-1** : Size of obtained solution  $F$  is  $\leq \alpha |F'|$ .

$$(G, k) \longrightarrow (G', k)$$

**Lemma-2** : Optimum does not increase. i.e.  $OPT(G', k) \leq OPT(G, k)$ .



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- Witness sets intersecting  $X$  and  $Y \setminus Y'$



**Reduction Rule 2 :** Given  $(G, k)$ , mark vertices in  $Y$  and delete unmarked vertices.

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**Lemma :** RR-2 is  $\alpha$  safe.



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**Lemma :** RR-2 is  $\alpha$  safe.

$$\frac{|F|}{\text{OPT}(G, k)} \leq \alpha \frac{|F'|}{\text{OPT}(G', k)}$$



**Theorem** : For any  $\alpha > 1$ , **CLIQUE CONTRACTION** parameterised by the size of solution  $k$  admits an  $\alpha$ -lossy kernel with  $O(k^{d+1})$  vertices where  $d = 1/\alpha$ .

**Theorem** : For any  $\alpha > 2$ , **SPLIT CONTRACTION** parameterised by the size of solution  $k$  admits an  $\alpha$ -lossy kernel with  $O(k^{\text{poly}(d)})$  vertices where  $d = 1/\alpha$ .

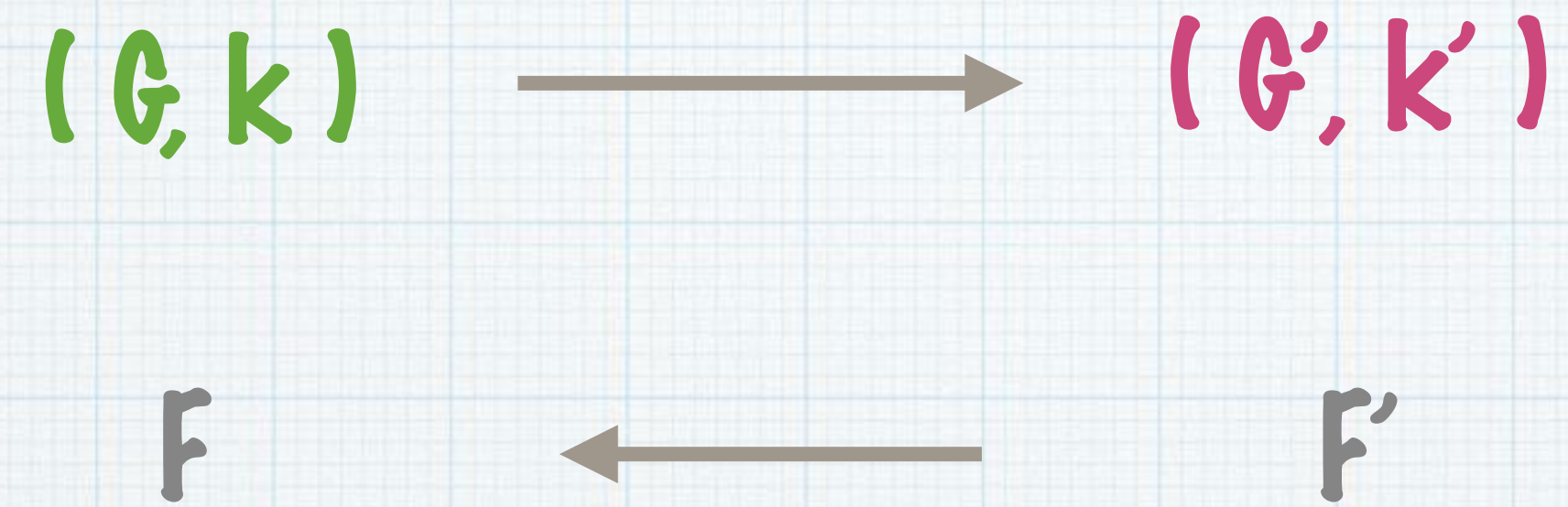


# SPLIT CONTRACTION

( Connecting  $\alpha$ -FPT approx algo with  $\alpha$ -lossy kernel )



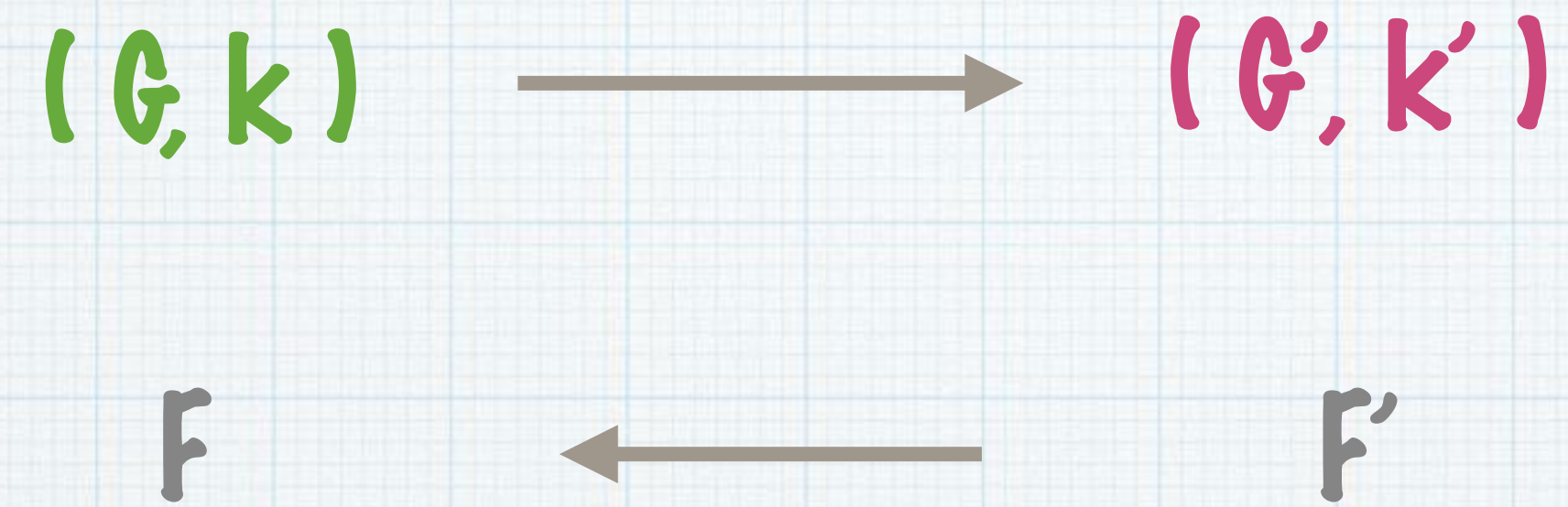
Assume  $\alpha$ -lossy kernel of size  $f(k)$



s.t. if  $F'$  is  $c$ -factor solution then  $F$  is  $(\alpha c)$ -factor solution.



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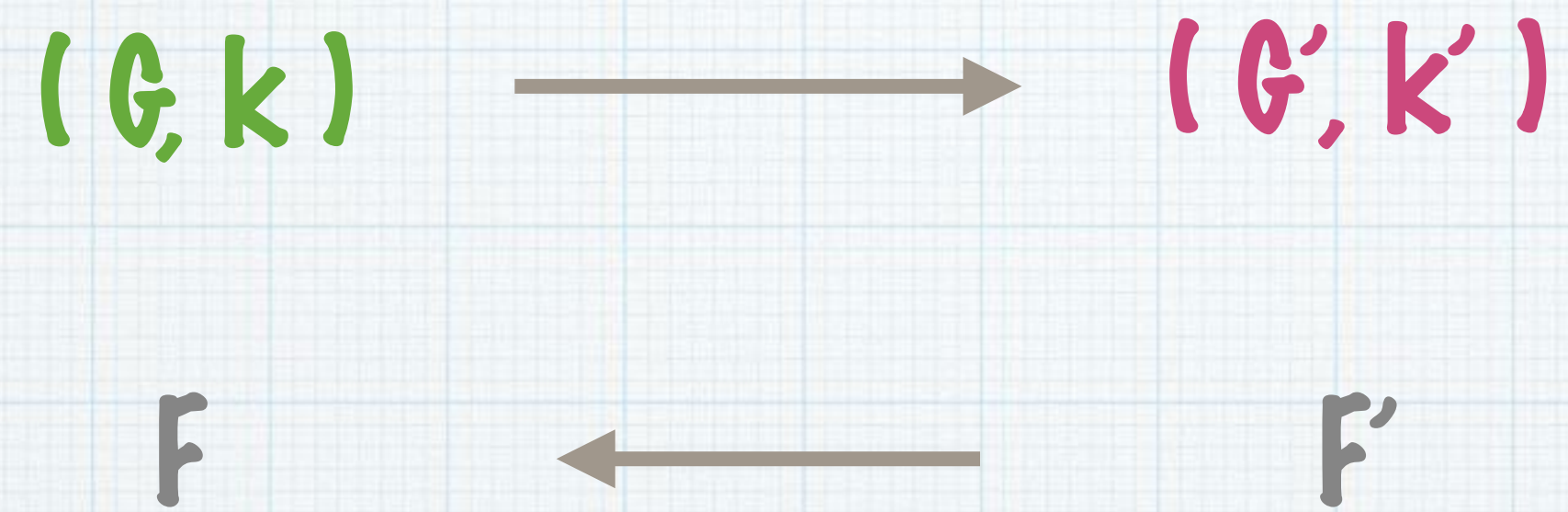


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Compute optimum solution for  $(G', k')$  in FPT time.



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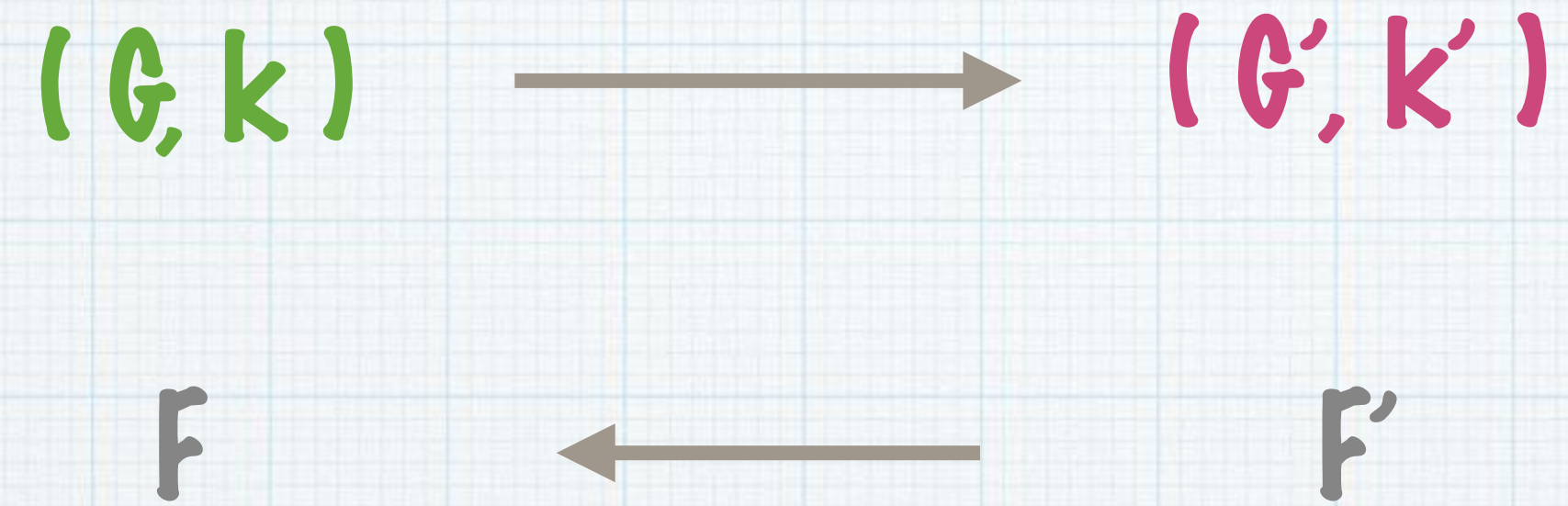
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s.t. if  $F'$  is  $c$ -factor solution then  $F$  is  $(\alpha c)$ -factor solution.

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Compute  $\alpha$ -factor solution for  $(G, k)$  in FPT time.

$\alpha$ -factor FPT approximation algorithm for SPLIT CONTRACTION.



Assume  $\alpha$ -lossy kernel of size  $f(k)$

$\alpha$ -factor FPT approximation algorithm for SPLIT CONTRACTION.

If there is a problem which

- does not admit an  $\alpha$ -factor FPT approximation algorithm
- has a gap preserving reduction to SPLIT CONTRACTION

then no  $\alpha$ -lossy kernel of any size exists.



## DENSEST $k$ SUBGRAPH

**Input** : Graph  $G$ , integers  $k, t$  and constants  $\epsilon < 1 < \beta$

**Parameters** :  $k + t$

**Output** : Are there at least  $\beta k$  vertices which spans  $\epsilon t$  many edges?

**Guarantee** : There is a cliques of size  $k$  in  $G$ .



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DENSEST  $k$  SUBGRAPH

$(G, k, t, \epsilon, \beta)$



SPLIT CONTRACTION

$(G', k')$



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— Runs in FPT time.



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- $G'$  can be contracted to a split graph with  $k'$  edge contractions (because of guarantee)



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SPLIT CONTRACTION

$(G', k')$

- Runs in FPT time.
- $G'$  can be contracted to a split graph with  $k'$  edge contractions (because of guarantee)
- For  $1.25 > \alpha$ , given an  $\alpha$ -factor solution for SPLIT CONTRACTION, we can obtain a solution for DENSEST  $k$  SUBGRAPH in poly-time.



**Theorem** : Assuming Gap-ETH, no FPT algorithm can approximate SPLIT CONTRACTION within a factor of  $\alpha$ , for any  $\alpha < 1.25$ .



# To sum up...

- \* Allowing only “edge contraction” makes Graph Modification Problems harder.
  - FPT algorithm
  - (classical) kernels
  - Exact algorithm.
- \* Can we obtain FPT approx algorithm and lossy kernels?



# Other successes

## \* Lossy kernels for

- TREE CONTRACTION (KMRT, FSTTCS 16)
- BOUNDED TREE CONTRACTION (ALST, CIAC 17)
- (Tree +  $q$  edges) CONTRACTION (AST, IPEC 17)
- CACTUS CONTRACTION (KRT, COCOON 18)

## \* Exact Algorithms

- PATH CONTRACTION (ADFST, ICALP 19)



# Open Problems regarding Contraction

- \* (No) kernels + lossy kernels

- Planar graphs
- Bi-partite graphs
- Outer-planar graphs
- Graphs of  $tw \leq 2$

- \* FPT algorithm for

- Interval graphs
- Outer-planar graphs
- Graph of  $tw \leq 2$

- \* Exact algorithms for

- Tree Contraction
- Clique Contraction \*



# Thank you.

( Any Questions? )