Romeo and Juliet Meeting in Forest like Regions

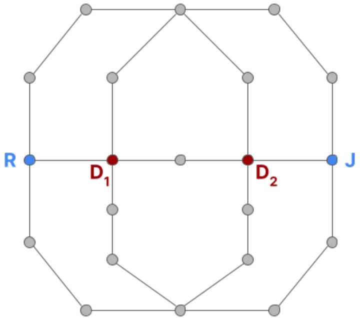
Neeldhara Misra[1] Manas Mulpuri[1] Prafullkumar Tale[2]
Gaurav Viramgami[1]

[1]Indian Institute of Technology, Gandhinagar, India

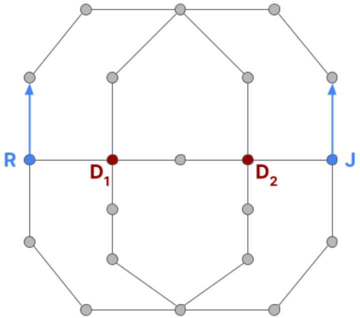
[2]Indian Institute of Science Education and Research, Pune, India

December 19, 2022

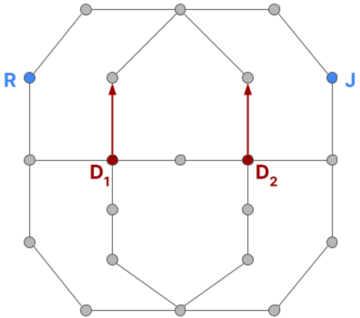
Rendezvous Game with Adversaries



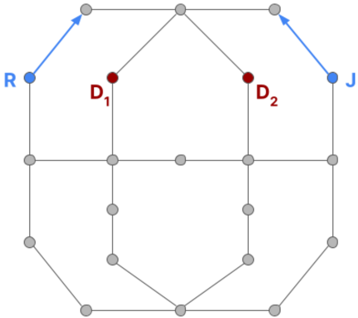
Rendezvous Game with Adversaries



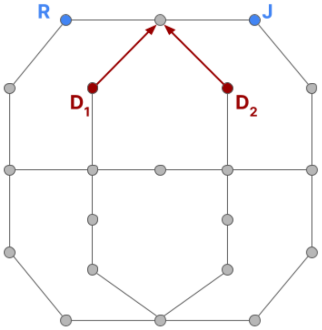
Rendezvous Game with Adversaries



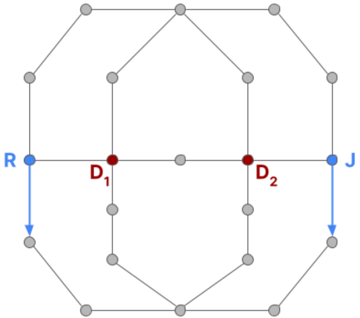
Rendezvous Game with Adversaries



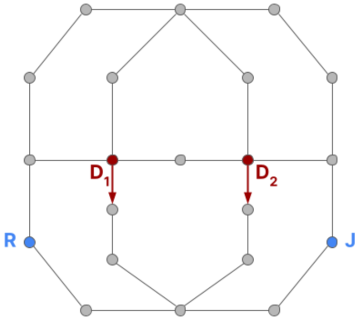
Rendezvous Game with Adversaries



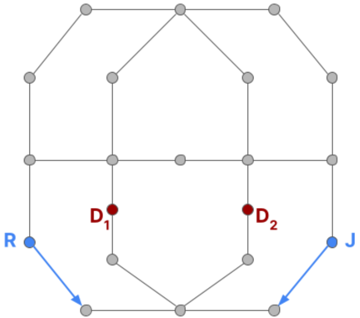
Rendezvous Game with Adversaries



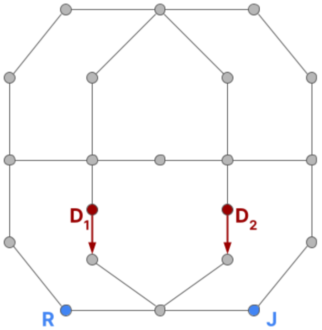
Rendezvous Game with Adversaries



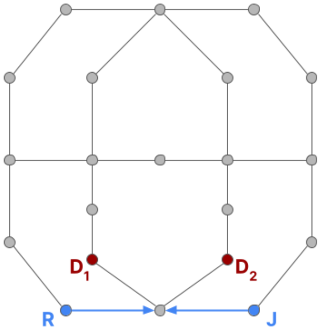
Rendezvous Game with Adversaries



Rendezvous Game with Adversaries



Rendezvous Game with Adversaries



Rendezvous Game with Adversaries

Fedor V. Fomin, Petr A. Golovach, and Dimitrios M. Thilikos.
Can romeo and juliet meet? or rendezvous games with adversaries on graphs.
Graph-Theoretic Concepts in Computer Science - 47th International Workshop, WG 2021

Problem Definition (RENDEZVOUS)

The game is played on a finite undirected connected graph G by two players: Facilitator and Divider.

Problem Definition (Rendezvous)

The game is played on a finite undirected connected graph G by two players: Facilitator and Divider.

- Facilitator has two agents Romeo and Juliet that are initially placed in designated vertices s and t of G.

Problem Definition (Rendezvous)

The game is played on a finite undirected connected graph G by two players: Facilitator and Divider.

- Facilitator has two agents Romeo and Juliet that are initially placed in designated vertices s and t of G.
- Divider has a team of $k \geq 1$ agents $D_1, \ldots, D_k$ that are initially placed in some vertices of $V(G) \setminus \{s, t\}$ chosen by him.

Problem Definition (Rendezvous)

The game is played on a finite undirected connected graph G by two players: Facilitator and Divider.

- Facilitator has two agents Romeo and Juliet that are initially placed in designated vertices s and t of G.
- Divider has a team of $k \geq 1$ agents $D_1, \ldots, D_k$ that are initially placed in some vertices of $V(G) \setminus \{s, t\}$ chosen by him.
- A single vertex can accommodate multiple agents of Divider.

Problem Definition (Rendezvous)

The game is played on a finite undirected connected graph G by two players: Facilitator and Divider.

- Facilitator has two agents Romeo and Juliet that are initially placed in designated vertices s and t of G.
- Divider has a team of $k \geq 1$ agents $D_1, \ldots, D_k$ that are initially placed in some vertices of $V(G) \setminus \{s, t\}$ chosen by him.
- A single vertex can accommodate multiple agents of Divider.
- The players make their moves by turn, starting with Facilitator. At every move, a player can move (*some of*) his/her agents to adjacent vertices unoccupied by adversary's agents.

Problem Definition (Rendezvous)

Rendezvous

Input: A graph G with two given vertices s and t, and a positive integer k. (s and t are distinct and not adjacent)
Question: Can Facilitator win on G starting from s and t against Divider with k agents?

Problem Definition (Rendezvous)

Rendezvous

Input: A graph G with two given vertices s and t, and a positive integer k. (s and t are distinct and not adjacent)
Question: Can Facilitator win on G starting from s and t against Divider with k agents?

Dynamic Separation: The dynamic separation number $d_G(s, t)$ is the minimum k such that Divider with k agents can win against Facilitator starting from s and t.

Parameterized Complexity

Let (I, k) be the given instance of problem Π, where k is the parameter.

Parameterized Complexity

Let (I, k) be the given instance of problem Π, where k is the parameter.

- FPT wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$

Parameterized Complexity

Let (I, k) be the given instance of problem Π, where k is the parameter.

- FPT wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$
- XP wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $|I|^{g(k)}$

Parameterized Complexity

Let (I, k) be the given instance of problem Π, where k is the parameter.

- FPT wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$
- XP wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $|I|^{g(k)}$
- para-NP-hard wrt k: Π is NP-hard even for constant k.

Parameterized Complexity

Let (I, k) be the given instance of problem Π, where k is the parameter.

- FPT wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$
- XP wrt k: We can decide whether (I, k) is a YES-INSTANCE of Π in time $|I|^{g(k)}$
- para-NP-hard wrt k: Π is NP-hard even for constant k.

Here, $f, g : N \rightarrow N$ are some computable functions depending only on k.

Main results shown by Fomin, Golovach, and Thilikos

- RENDEZVOUS is PSPACE-hard and, when parameterized by k, co-W[2]-hard.

Main results shown by Fomin, Golovach, and Thilikos

- Rendezvous is PSPACE-hard and, when parameterized by k, co-W[2]-hard.
- $|V(G)|^{\mathcal{O}(k)}$ time algorithm for Rendezvous based on backtracking stages over the game arena.

Main results shown by Fomin, Golovach, and Thilikos

- Rendezvous is PSPACE-hard and, when parameterized by k, co-W[2]-hard.
- $|V(G)|^{\mathcal{O}(k)}$ time algorithm for Rendezvous based on backtracking stages over the game arena.
- Rendezvous admits polynomial time algorithms on chordal graphs and P_5-free graphs.

Our Contributions

Are there other natural graph classes on which RENDEZVOUS admits polynomial time algorithm?

Our Contributions

Are there other natural graph classes on which RENDEZVOUS admits polynomial time algorithm?

- We present a polynomial time algorithm for RENDEZVOUS on grid graphs and treewidth at most 2, i.e. series-parallel graphs.

Our Contributions

Is RENDEZVOUS parameterized by vertex cover number?

Our Contributions

Is RENDEZVOUS parameterized by vertex cover number?

- We present a natural exponential kernel in the combined parameter vertex cover and solution size k, and hence FPT in the combined parameter.

Our Contributions

Is RENDEZVOUS parameterized by vertex cover number?

- We present a natural exponential kernel in the combined parameter vertex cover and solution size k, and hence FPT in the combined parameter.
- This kernel cannot be improved to a polynomial kernel under standard complexity-theoretic assumptions.

Our Contributions

Is RENDEZVOUS FPT or XP parameterized by treewidth?

Our Contributions

Is RENDEZVOUS FPT or XP parameterized by treewidth?

- We answer this question in the negative by showing that RENDEZVOUS is in fact co-NP-hard even for graphs of constant treewidth.

Our Contributions

Is RENDEZVOUS FPT or XP parameterized by treewidth?

- We answer this question in the negative by showing that RENDEZVOUS is in fact co-NP-hard even for graphs of constant treewidth.
- Specifically, we show that RENDEZVOUS is co-NP-hard even when restricted to:
  - Graphs whose feedback vertex set number is at most 14 or
  - Graphs whose pathwidth is at most 16

Our Contributions

Is RENDEZVOUS FPT or XP parameterized by treewidth?

- We answer this question in the negative by showing that RENDEZVOUS is in fact co-NP-hard even for graphs of constant treewidth.
- Specifically, we show that RENDEZVOUS is co-NP-hard even when restricted to:
  - Graphs whose feedback vertex set number is at most 14 or
  - Graphs whose pathwidth is at most 16
- We also show that the problem is unlikely to admit an FPT algorithm even when parameterized by the combined parameters FVS $+ k$ or pathwidth $+ k$.

Grid Graphs

Theorem: RENDEZVOUS *can be solved in polynomial time on grid graphs.*

Grid Graphs

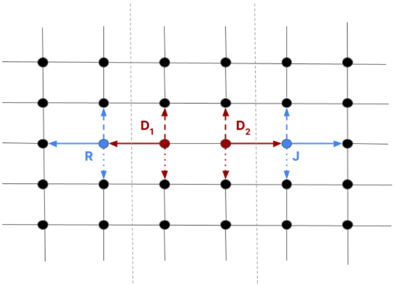
Theorem: RENDEZVOUS *can be solved in polynomial time on grid graphs.*

Observation: On grid graphs, $d_G(s, t) = 2$.

Grid Graphs

Theorem: RENDEZVOUS *can be solved in polynomial time on grid graphs.*

Observation: On grid graphs, $d_G(s, t) = 2$.

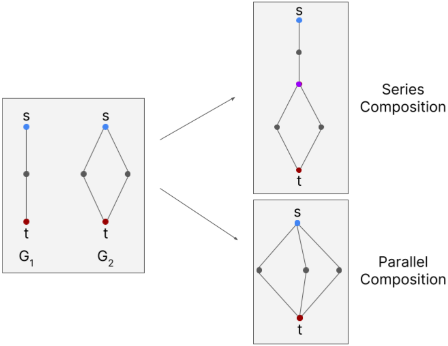


Graphs with treewidth at most 2 [Series-Parallel Graphs]

Series–Parallel graphs are graphs with two distinguished vertices called *terminals*, formed recursively by two simple composition operations.

Graphs with treewidth at most 2 [Series-Parallel Graphs]

Series–Parallel graphs are graphs with two distinguished vertices called *terminals*, formed recursively by two simple composition operations.



Graphs with treewidth at most 2 [Series Parallel Graphs]

Theorem: RENDEZVOUS *can be solved in polynomial time on graphs with treewidth at most* 2*, i.e. series-parallel graphs.*

Graphs with treewidth at most 2 [Series Parallel Graphs]

Theorem: RENDEZVOUS *can be solved in polynomial time on graphs with treewidth at most* 2, *i.e. series-parallel graphs.*

Idea of the proof: By proving dynamic separation number to be equal to static separation number; applying induction on the number of vertices and case analysis for sequence of compositions used to arrive at the final graph G.

Graphs with treewidth at most 2 [Series Parallel Graphs]

Theorem: RENDEZVOUS *can be solved in polynomial time on graphs with treewidth at most* 2, *i.e. series-parallel graphs.*

Idea of the proof: By proving dynamic separation number to be equal to static separation number; applying induction on the number of vertices and case analysis for sequence of compositions used to arrive at the final graph G.

Static separation number can be computed in polynomial time.

FPT Parameterized by Vertex Cover & Solution Size

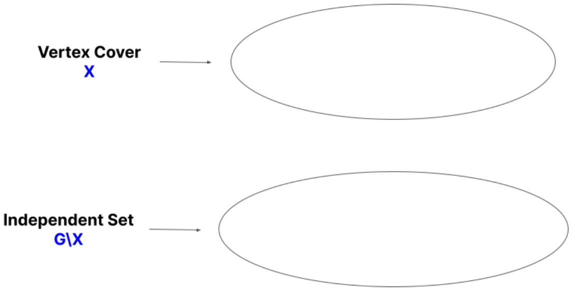
Theorem: RENDEZVOUS *is FPT when parameterized by the vertex cover number of the input graph and the solution size.*

FPT Parameterized by Vertex Cover & Solution Size

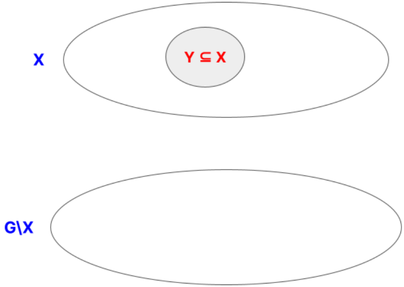
Theorem: RENDEZVOUS *is FPT when parameterized by the vertex cover number of the input graph and the solution size.*

Idea of the proof: We present a natural exponential kernel in the combined parameter vertex cover and solution size k, and the theorem follows from the $|V(G)|^{\mathcal{O}(k)}$ time algorithm for RENDEZVOUS given by Fomin, Golovach, and Thilikos.

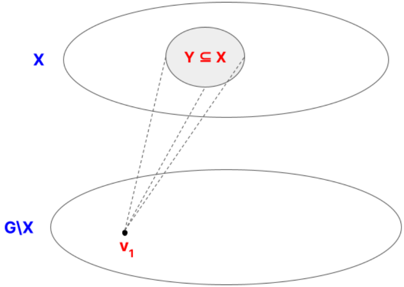
Sketch of the Proof (Reduction Rule)



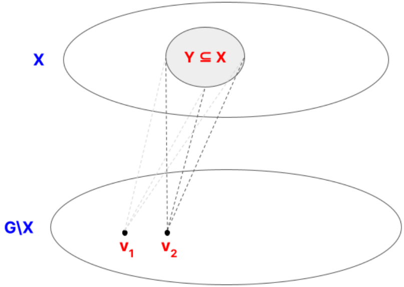
Sketch of the Proof (Reduction Rule)



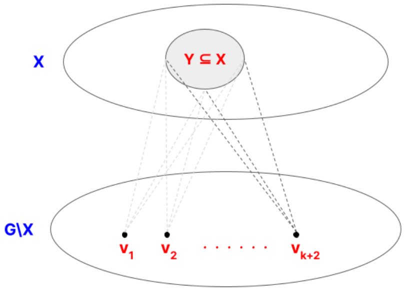
Sketch of the Proof (Reduction Rule)



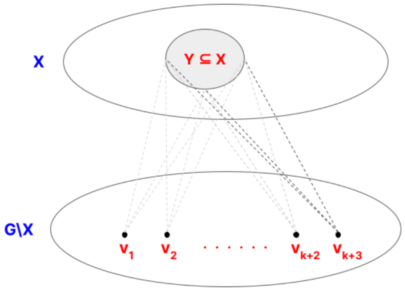
Sketch of the Proof (Reduction Rule)



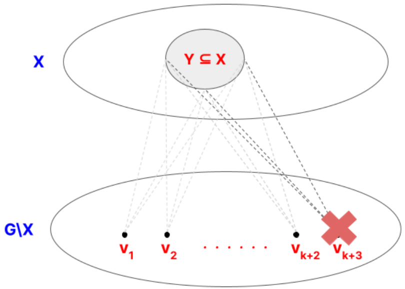
Sketch of the Proof (Reduction Rule)



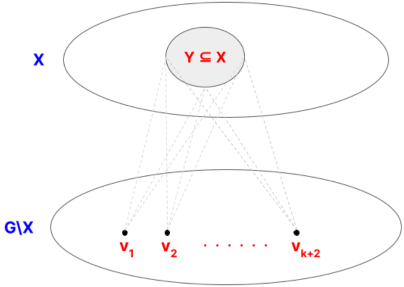
Sketch of the Proof (Reduction Rule)



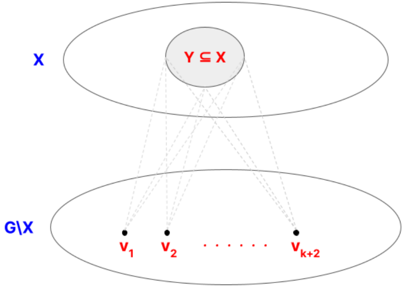
Sketch of the Proof (Reduction Rule)



Sketch of the Proof (Reduction Rule)



Sketch of the Proof (Reduction Rule)



$\forall Y \subseteq X$, at max $k+2$ vertices will be connected to $Y \implies$
$|V(G \backslash X)| \leq 2^{|X|} \cdot (k+2) \implies |V(G)| \leq |VC| + 2^{|VC|} \cdot (k+2)$

co-para-NP-hardness parameterized by FVS

Theorem: RENDEZVOUS *is co-NP-hard even when restricted to graphs whose feedback vertex set number is at most* 14.

[1] One of the Karp's 21 NP-complete problems

co-para-NP-hardness parameterized by FVS

Theorem: RENDEZVOUS *is co-NP-hard even when restricted to graphs whose feedback vertex set number is at most* 14.

Idea of the proof: Reduction from the 3-DIMENSIONAL MATCHING[1] problem.

[1] One of the Karp's 21 NP-complete problems

co-para-NP-hardness parameterized by FVS

Theorem: RENDEZVOUS *is co-NP-hard even when restricted to graphs whose feedback vertex set number is at most* 14*.*

Idea of the proof: Reduction from the 3-DIMENSIONAL MATCHING[1] problem.

The same reduction also implies that, RENDEZVOUS is co-NP-hard even when restricted to graphs whose pathwidth is at most 16.

[1] One of the Karp's 21 NP-complete problems

co-W[1]-hardness parameterized by FVS & Solution Size

Theorem: RENDEZVOUS *is co-W[1]-hard when parameterized by the feedback vertex set number and the solution size.*

[2]Shown W[1]-hard when parameterized by number of variables by K. Bringmann, D. Hermelin, M. Mnich, and E. J. van Leeuwen

co-W[1]-hardness parameterized by FVS & Solution Size

Theorem: RENDEZVOUS *is co-W[1]-hard when parameterized by the feedback vertex set number and the solution size.*

Idea of the proof: Reduction from the (MONOTONE) NAE-INTEGER-3-SAT[2] problem.

[2]Shown W[1]-hard when parameterized by number of variables by K. Bringmann, D. Hermelin, M. Mnich, and E. J. van Leeuwen

co-W[1]-hardness parameterized by FVS & Solution Size

Theorem: RENDEZVOUS *is co-W[1]-hard when parameterized by the feedback vertex set number and the solution size.*

Idea of the proof: Reduction from the (MONOTONE) NAE-INTEGER-3-SAT[2] problem.

The same reduction also implies that, RENDEZVOUS is co-W[1]-hard when parameterized by the pathwidth and the solution size.

[2]Shown W[1]-hard when parameterized by number of variables by K. Bringmann, D. Hermelin, M. Mnich, and E. J. van Leeuwen

3-Dimensional Matching

3-Dimensional Matching

Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?

3-Dimensional Matching

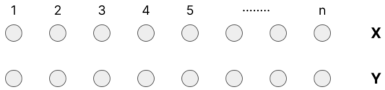
3-Dimensional Matching

Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?

3-Dimensional Matching

3-Dimensional Matching

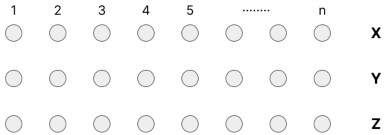
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

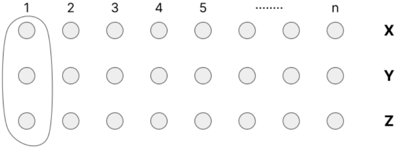
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

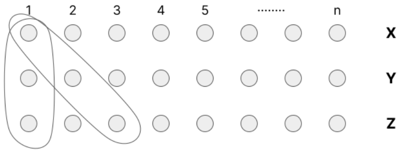
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

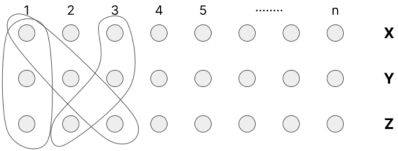
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

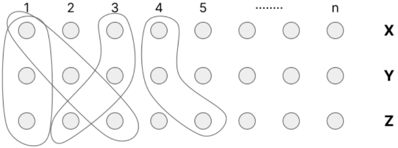
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

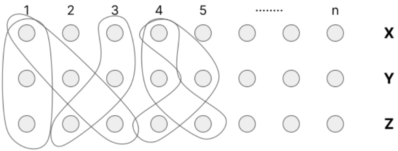
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

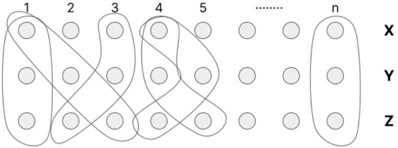
Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



3-Dimensional Matching

3-Dimensional Matching

Input: Sets X, Y, Z each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.
Question: Is there a set of n triplets in T such that each element is contained in exactly one triplet?



(Monotone) NAE-Integer-3-Sat

(Monotone) NAE-Integer-3-Sat

Input: Variables $x_1, \ldots, x_k$ that each take a value in $\{1, \ldots, n\}$ and clauses $C_1, \ldots, C_m$ of the form $NAE(x_{i_1} \leq a_1, x_{i_2} \leq a_2, x_{i_3} \leq a_3), a_1, a_2, a_3 \in \{1, \ldots, n\}$, which is satisfied if not all three inequalities are true and not all are false (i.e., they are "not all equal")

Question: Is there exists an assignment of the variables that satisfies all given clauses?

Open Problem

- Is RENDEZVOUS W[1]-hard when parameterized by the vertex cover number (only)?

Open Problem

- Is RENDEZVOUS W[1]-hard when parameterized by the vertex cover number (only)?

Thank You!