Reducing the Vertex Cover Number via Edge Contractions

Uéverton Souza
Universidade Federal Fluminense, Brazil
University of Warsaw, Poland

Paloma Lima[a], Vinicius F. Santos[b], Ignasi Sau[c], Prafullkumar Tale[d]
[a]IT University of Copenhagen, Denmark
[b]Universidade Federal de Minas Gerais, Brazil
[c]Université de Montpellier, France
[d]CISPA Helmholtz Center for Information Security, Germany

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Blocker problems

Let π be a graph parameter
(independence number, domination number, size of longest path, ...).

Let $\mathcal{M}$ be a set of allowed graph modification operations
(vertex deletion, edge deletion/addition/contraction, ...).

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Input:	A graph G and two integers k, d.
Question:	Can G be modified into a graph G', via at most k operations from $\mathcal{M}$, such that $\pi(G') \leq \pi(G) - d$?

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- π = chromatic/independence/clique/matching/domination number
  [Bentz et al. 2010] [Costa et al. 2011] [Bazgan et al. 2011, 2015]
  [Diner et al. 2018] [Paulusma et al. 2019] [Fomin et al. 2020]

Particular case: edge contractions

We focus on $\mathcal{M} = \{$edge contraction$\}$.

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- π = feedback vertex set/odd cycle transversal/vertex cover number

[Lima, Santos, Sau, Souza, JCSS 2021]

In the previous work [Lima, Santos, Sau, Souza, JCSS 2021]

Corollary

Contraction(fvs), Contraction(oct) *are* co-NP-*hard for* $k = d = 1$.

Is the Contraction(π) problem always hard for natural parameters π?

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[Krithika et al. 2016]

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- Star Contraction $\equiv$ Connected Vertex Cover.

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- Connected Vertex Cover is NP-hard even if vc is polynomial (bipartite graphs). [Escoffier et al. 2010]

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Rank of a graph

rank(G) is the number of vertices of G minus its number of connected components (or equivalently, the number of edges of a set of spanning trees of each of the connected components of G).

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To decide whether an instance (G, k, d) *of* CONTRACTION(vc) *is a* YES*-instance is*

- *coNP-hard if* $k = rank(G)$,
- *coNP-hard if* $k < rank(G)$ *and* $2d \leq k$, *and*
- *NP-hard if* $k < rank(G)$ *and* $k = d$.

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- If G is connected, $k < rank(G)$, and $2d \leq k$, then (G, k, d) is a Yes-instance if and only if $d < vc(G)$.

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- Reduction from MULTICOLORED INDEPENDENT SET.

An improved algorithm

Theorem [Lima, Santos, Sau, Souza, JCSS 2021]

The CONTRACTION(vc) problem can be solved on n-vertex graphs in time $f(d) \cdot n^{2d}$ for some computable function f.

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Theorem

There exists an algorithm that solves $\textsc{Contraction}(\textsc{vc})$ *in time*

$$2^{\mathcal{O}(d)} \cdot n^{k-d+\mathcal{O}(1)}.$$

In particular, the problem is FPT parameterized by d when $k = d$.

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Theorem

Contraction(vc) *is W[1]-hard parameterized by* $k + d$*. Moreover, unless the ETH fails, it does not admit an algorithm running in time* $f(k+d) \cdot n^{o(k+d)}$ *for any computable function* $f : \mathbb{N} \mapsto \mathbb{N}$.

The result holds even if G is a bipartite graph with a bipartition $\langle X, Y \rangle$ such that X is a minimum vertex cover of G.

W[1]-hardness parameterized by $k + d$

Edge Induced Forest (EIF)

Given a graph G and an integer ℓ, the goal is to determine whether G has a set F of at least ℓ edges such that $G[V(F)]$ is a forest?

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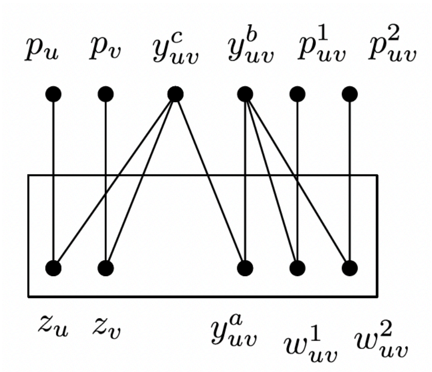
EIF *parameterized by the size of the solution* ℓ*, is W[1]-hard. Moreover, unless the ETH fails, it does not admit an algorithm running in time* $f(\ell) \cdot n^{o(\ell)}$ *for any computable function* $f : \mathbb{N} \mapsto \mathbb{N}$.

Reduction from EIF parameterized by ℓ



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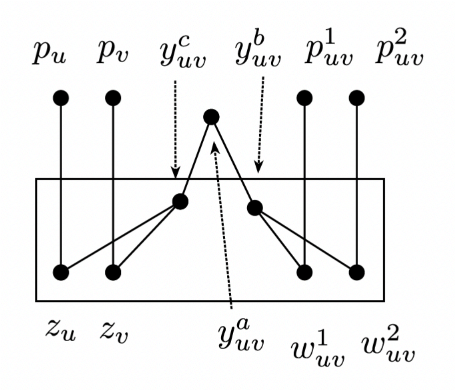

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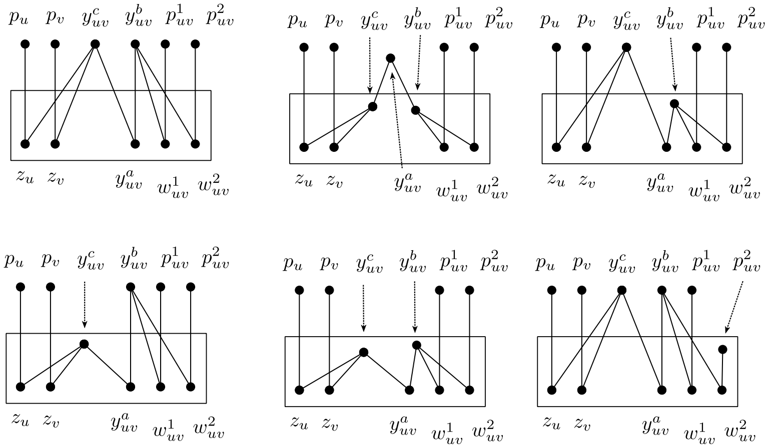

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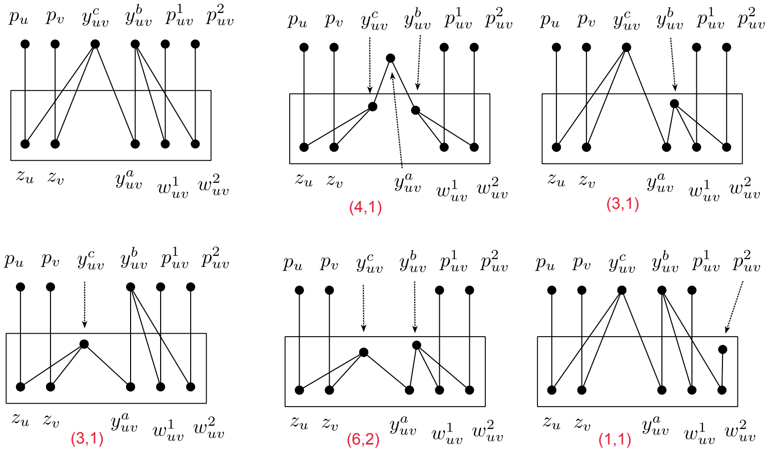

Summary of our main results on Contraction(vc)

- NP-hardness for $k = d$.
- A $(2^{\mathcal{O}(d)} \cdot n^{k-d+\mathcal{O}(1)})$-time algorithm.
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Thanks!