Parameterized Subexponential Algorithms for Generalized Cycle Hitting Problems on Planar Graphs

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MPI Noon Seminar

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- ideas often problem-specific

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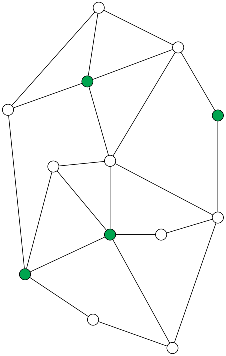

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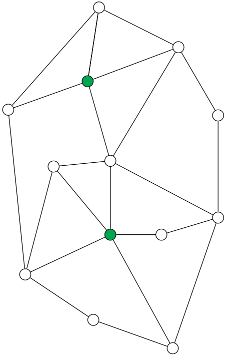

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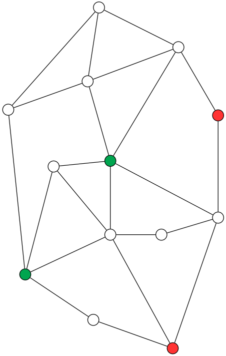

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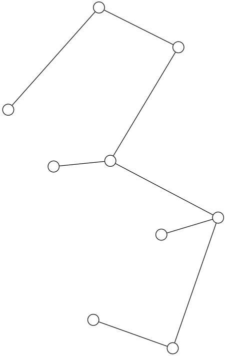

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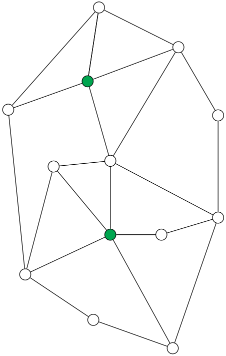

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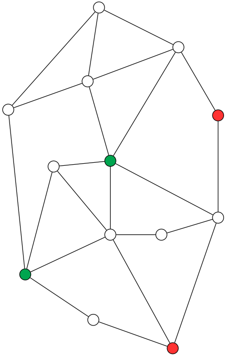

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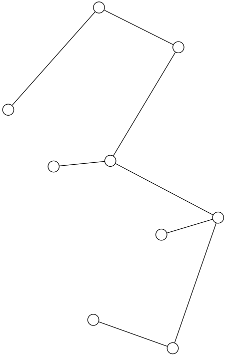

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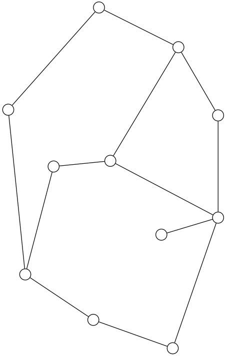

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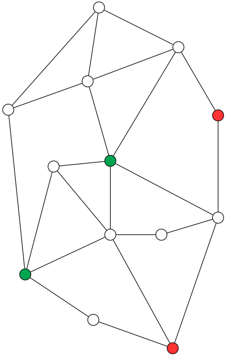

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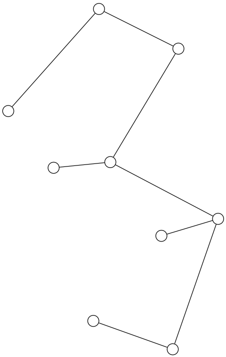

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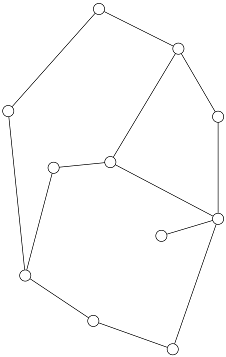

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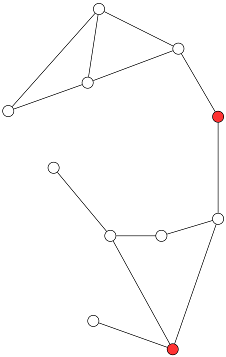

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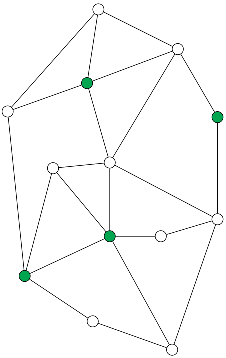

Bidimensionality
[Demaine et al.]

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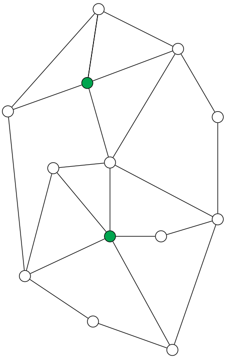

Extended Contraction Decomposition

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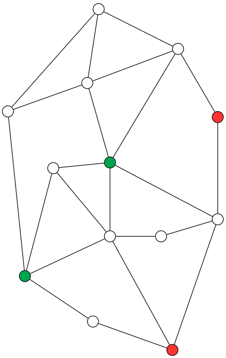

Extended Contraction Decomposition & Body Guessing

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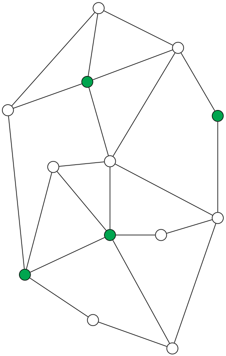

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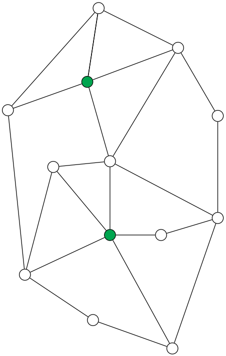

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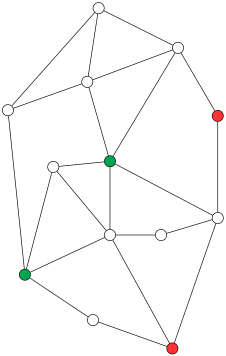

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Feedback Vertex Set

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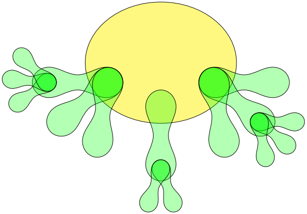

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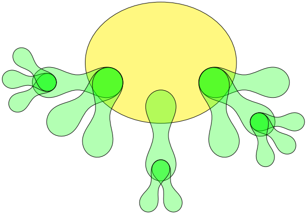
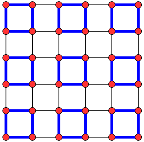


Grid Minor Theorem



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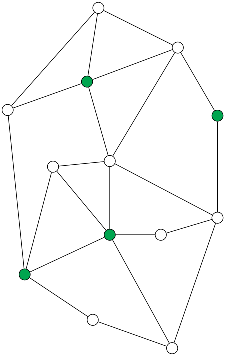

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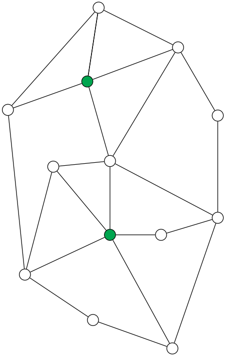

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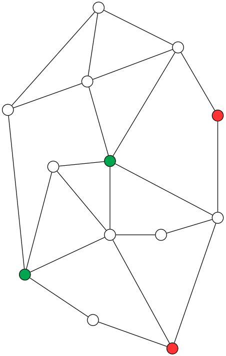

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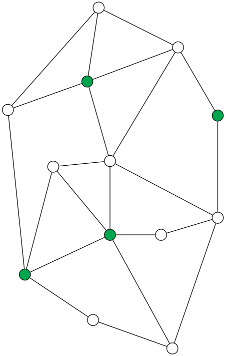

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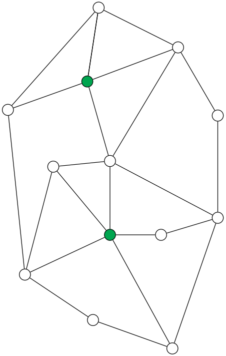

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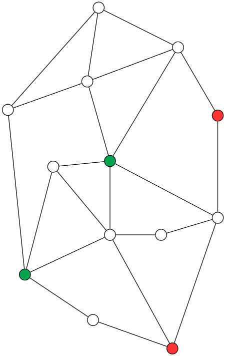

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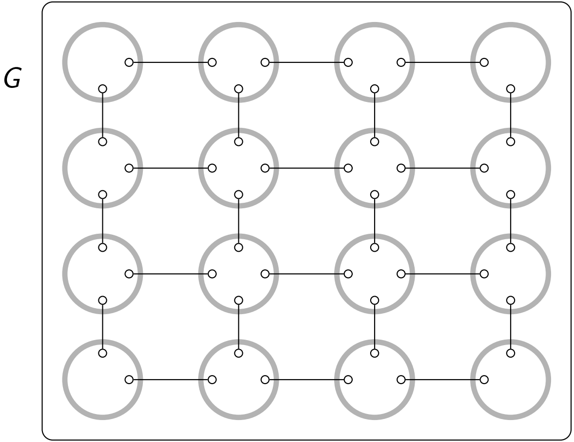

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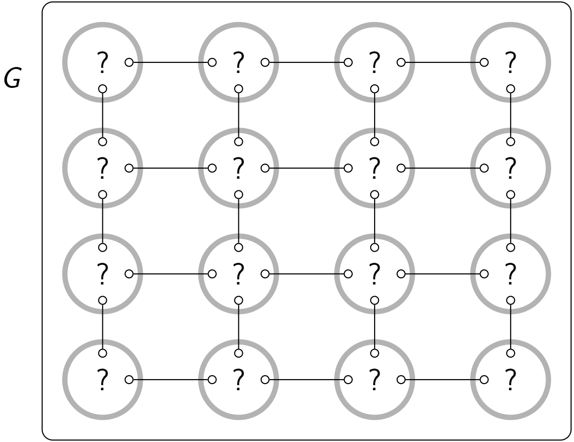

A Slightly Extended Task

Relaxed Goal: Find XP-algorithm running in time $n^{\mathcal{O}(\sqrt{k})}$

Extended OCT

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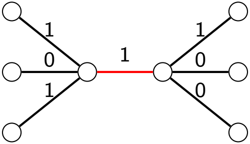

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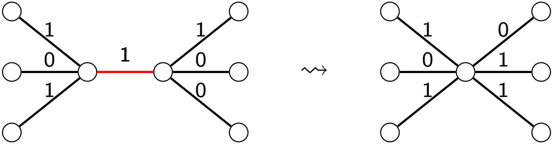

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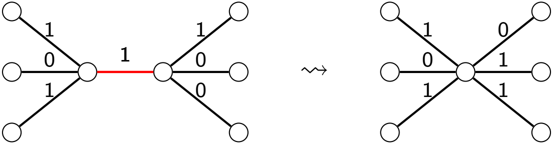
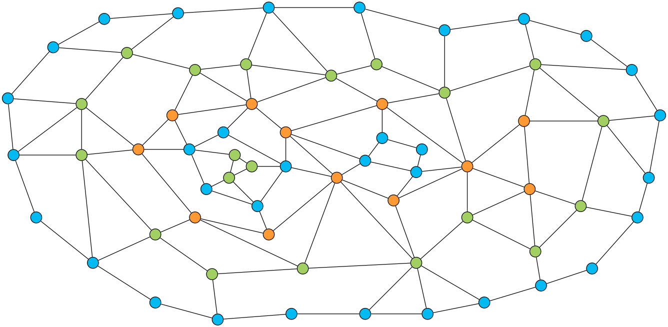


Idea: Find edges that can be safely contracted until the tree-width is small.

Extended Contraction Decompositions



Theorem

Let G be a planar graph and $\ell \geq 1$. Then there is a polynomial-time computable partition $V(G) = V_1 \uplus \cdots \uplus V_\ell$ such that

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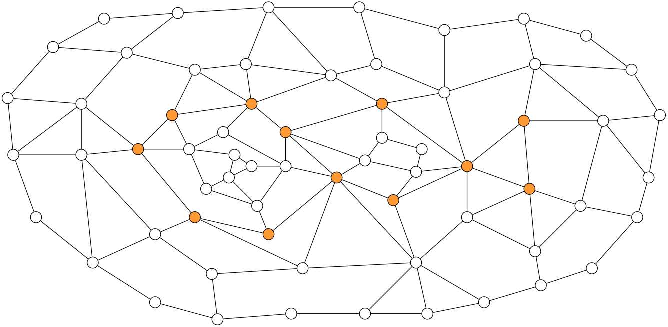

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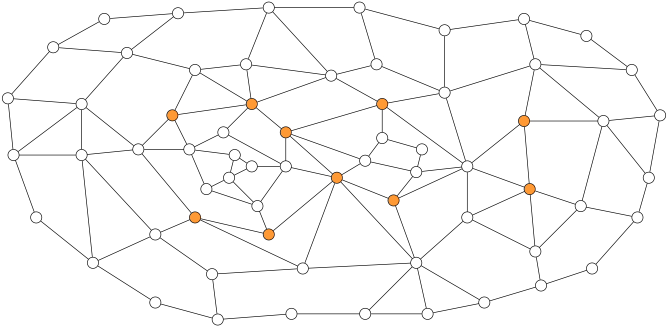

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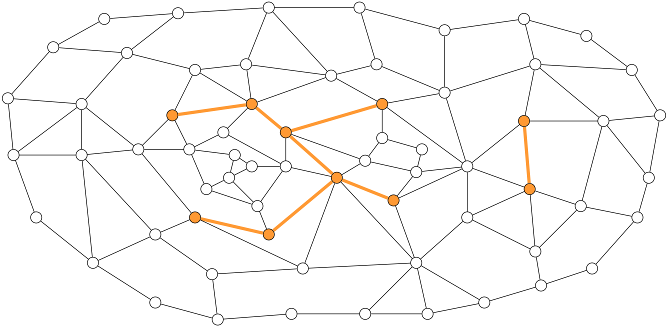

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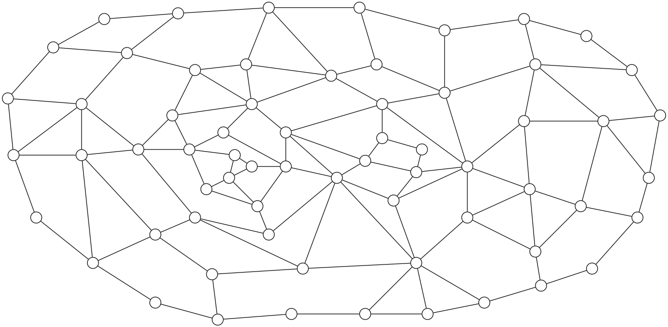

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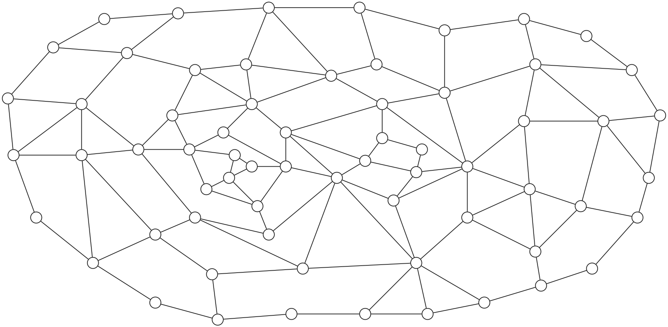

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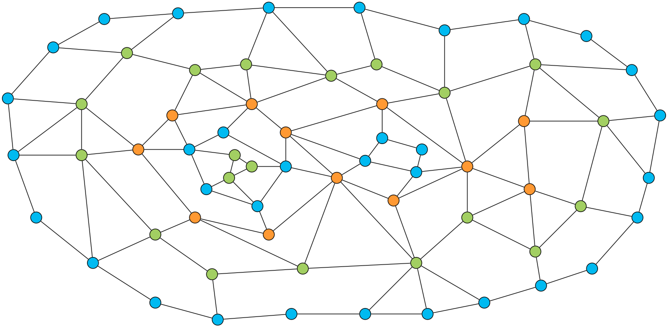

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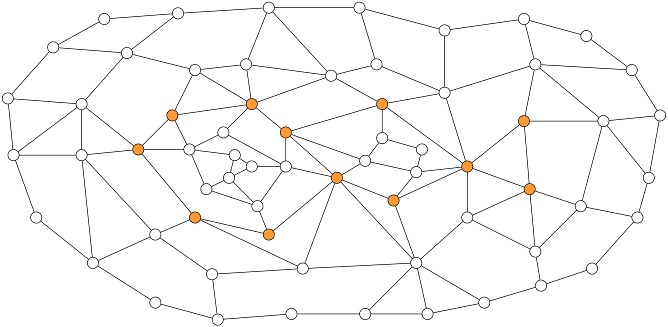

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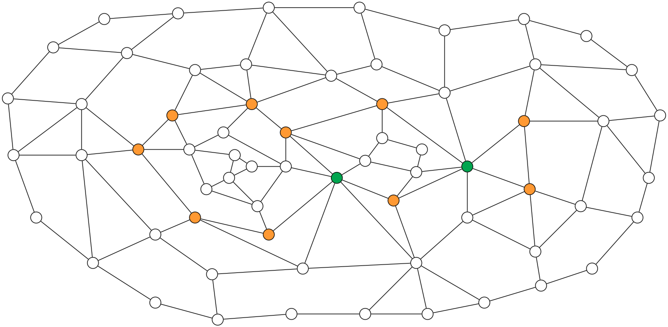

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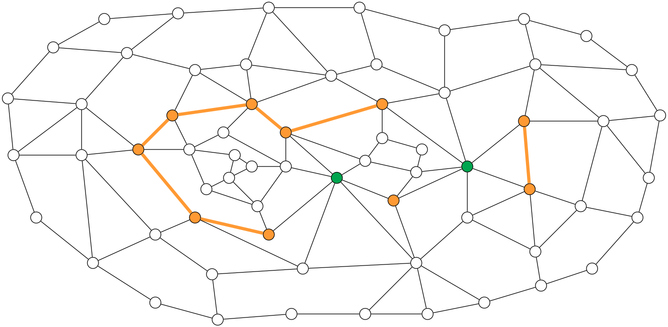

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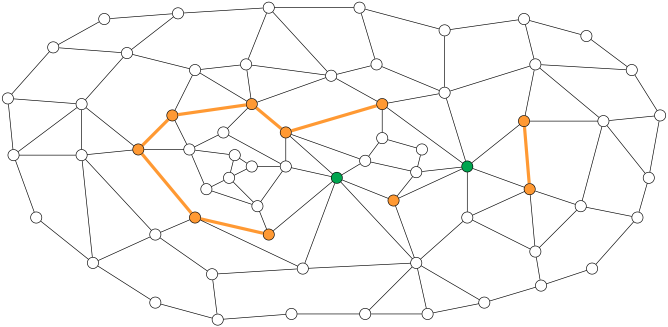

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5. contract all edges between vertices from $V_i \setminus Z_i$ — $\mathcal{O}(n)$
6. solve problem by DP on graph of tree-width $\mathcal{O}(\ell + |Z_i|)$ — $2^{\mathcal{O}(\sqrt{k})} n$

Exploiting Kernels

A parameterized problem Π *admits a polynomial kernel* if there is a polynomial-time algorithm that, given an instance (I, k) of Π, computes an equivalent instance (I', k') of Π such that $k' = k^{\mathcal{O}(1)}$ and $|I'| = k^{\mathcal{O}(1)}$.

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Theorem

Planar OCT parameterized by solution size admits a randomized polynomial compression into Planar OCT with undeletable vertices that does not increase the parameter.

Corollary (Lokshtanov et al. 2012)

Planar OCT can be solved in randomized time $2^{\widetilde{\mathcal{O}}(\sqrt{k})} n^{\mathcal{O}(1)}$.

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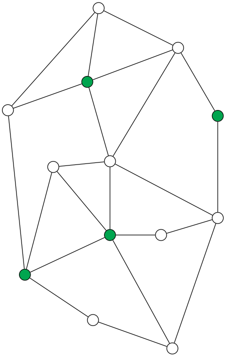

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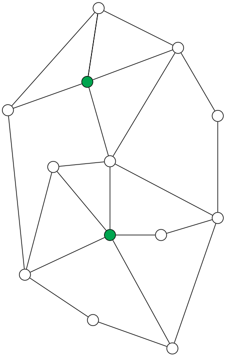
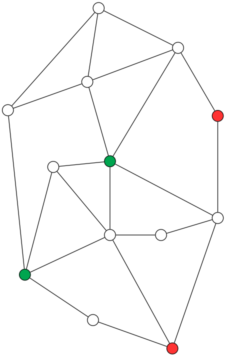


Extended Contraction Decomposition

Subset-OCT

Hit all odd cycles through T

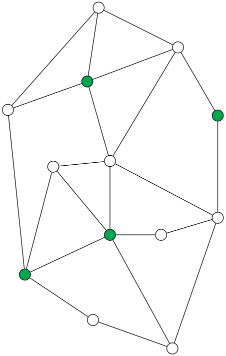


Extended Contraction Decomposition & Body Guessing

Techniques

FVS

Hit all cycles



Bidimensionality

[Demaine et al.]

OCT

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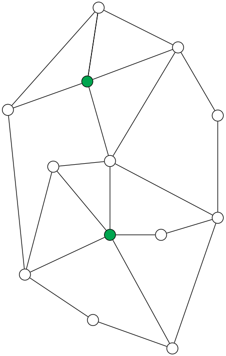

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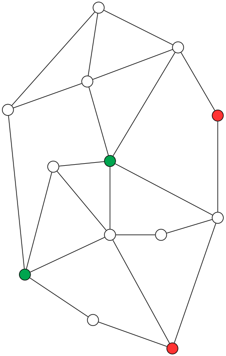

Extended Contraction Decomposition & Body Guessing

The Next Level: Problems on Two-Connected Components

Subset Odd Cycle Transversal (Subset OCT)

Input: Graph G, a set $T \subseteq V(G)$, and integer k

Task: Find $Z \subseteq V(G)$ of size $\leq k$ such that $G - Z$ has no odd T-cycle

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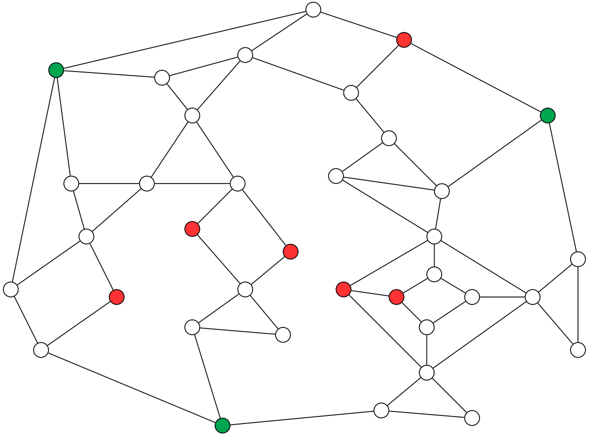

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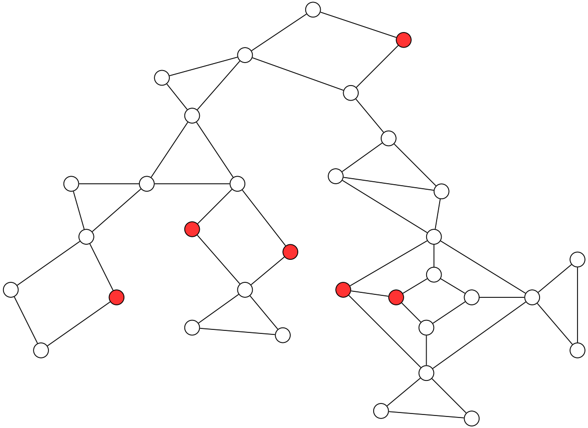

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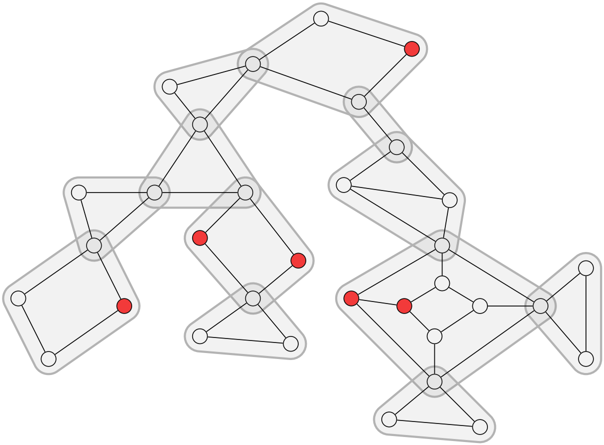

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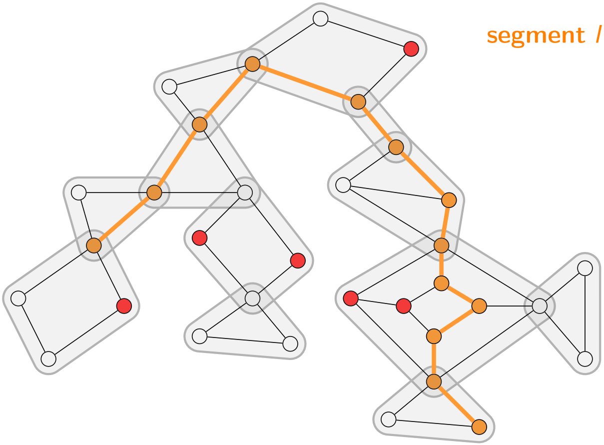

segment / to be contracted

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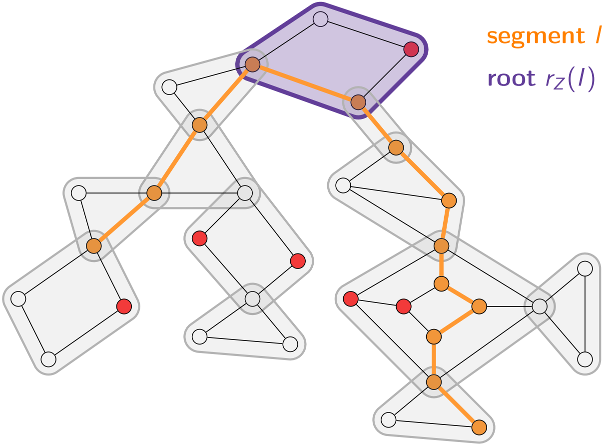

segment I to be contracted

root $r_Z(I)$ of I

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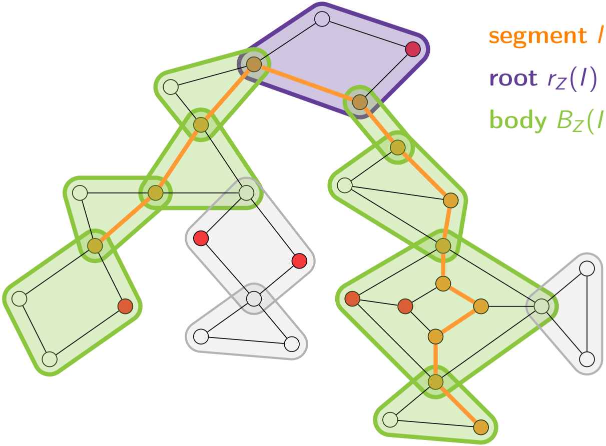

segment I to be contracted

root $r_Z(I)$ of I

body $B_Z(I)$ of I

Contraction Decomposition with Body Guessing

Theorem

Given a planar graph and an integer k, one can compute in time $n^{\mathcal{O}(\sqrt{k})}$ a sequence $(V_1, \mathcal{B}_1), \ldots, (V_h, \mathcal{B}_h)$ with $h = n^{\mathcal{O}(\sqrt{k})}$, where $V_i \subseteq V(G)$ and $\mathcal{B}_i$ contains $n^{\mathcal{O}(1)}$ sets of vertices, such that the following holds. For every $Z \subseteq V(G)$ with $|Z| \leq k$, there is at least one $1 \leq i \leq h$ such that

1. $V_i \cap Z = \emptyset$,
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- since planar graphs have constant average degree, one of those i provides segments whose bodies have small interaction with solution vertices

Subexponential Algorithms

Theorem

Planar Subset OCT can be solved in time $n^{\mathcal{O}(\sqrt{k})}$.

Subexponential Algorithms

Theorem

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Subset Feedback Vertex Set (Subset FVS)

Input: Graph G, a set $T \subseteq V(G)$, and integer k
Task: Find $Z \subseteq V(G)$ of size $\leq k$ such that $G - Z$ has no T-cycle

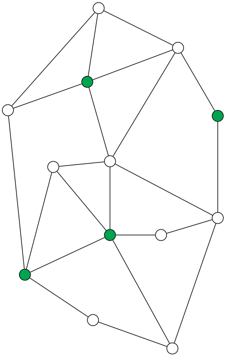
Theorem

Planar Subset FVS can be solved in randomized time $2^{\widetilde{\mathcal{O}}(\sqrt{k})} n^{\mathcal{O}(1)}$.

Techniques

FVS

Hit all cycles



Bidimensionality

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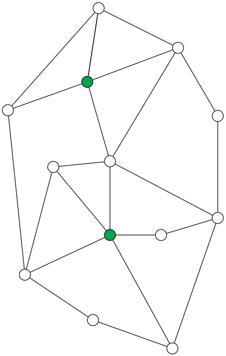
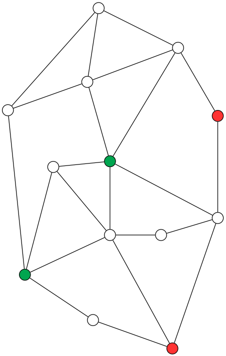


Extended Contraction Decomposition

Subset-OCT

Hit all odd cycles through T



Extended Contraction Decomposition & Body Guessing

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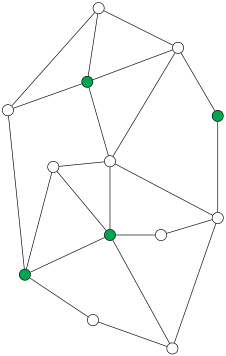

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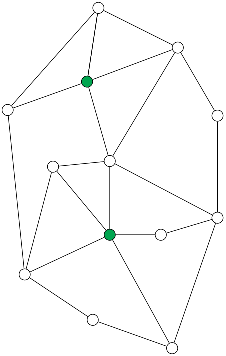

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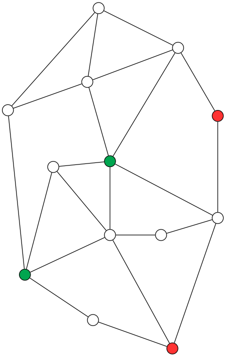

Extended Contraction Decomposition & Body Guessing

Wrap-Up - Extensions

CSPs and Size Constraints:

- techniques extend to Permutation CSP Deletion problems
- size bounds for global appearances, or in connected or two-connected components

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***H*-Minor-Free Graphs:**

- almost all arguments carry over to H-minor-free graphs for fixed H
- only missing piece is the Extended Contraction Decomposition
- we conjecture that it also holds for H-minor-free graphs
- proved by Bandyapadhyay et. al (SODA '22) for h-alomst embeddable graphs