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Parameterized Complexity of Weighted Multicut in Trees

WG 2022

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June 23, 2022

Definition of Multicut

Input: An undirected graph G, vertex pairs $(s_1, t_1), \ldots, (s_p, t_p) \in V(G) \times V(G)$, and a positive integer k.
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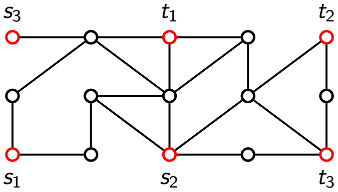

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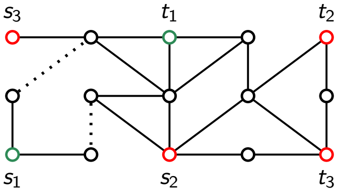

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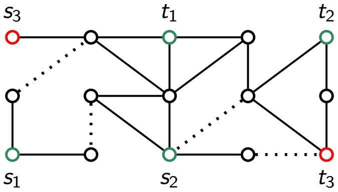

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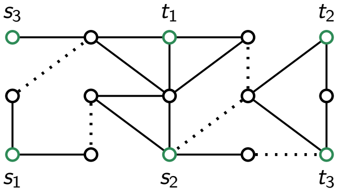

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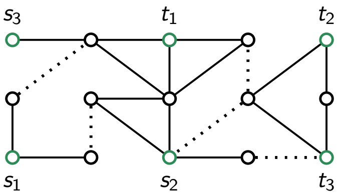

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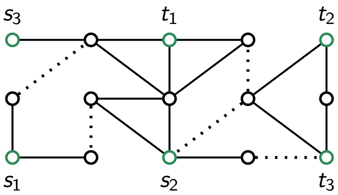

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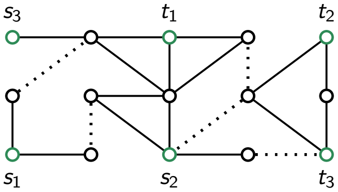

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Simple branching algorithm (Guo and Niedermeier '05):

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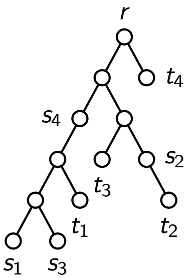

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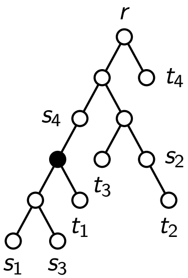

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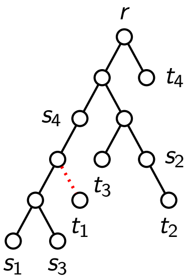

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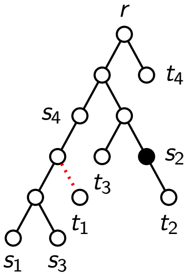

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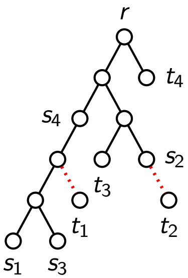

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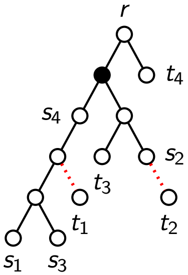

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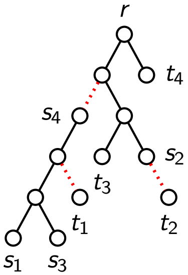

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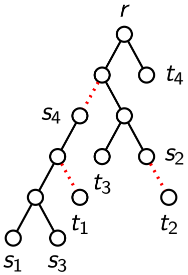

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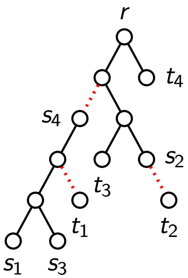

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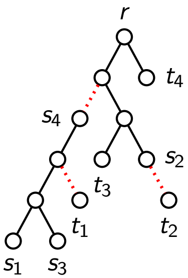

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A problem with running time $f(k) \cdot n^{\mathcal{O}(1)}$ is *fixed parameter tractable* (FPT).
FPT also denotes the class of "efficient" problems in the parameterized setting.

Adding Weights

As for other problems keep size constraint and add weight constraint.

- Weighted (s, t)-Cut
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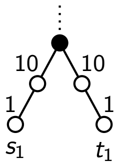

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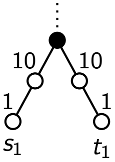

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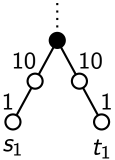
$\implies$ Focus on (subdivided) stars



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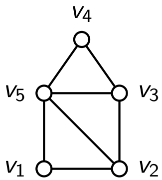

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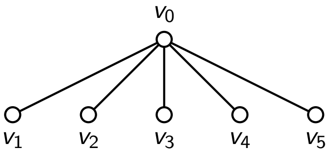

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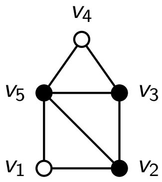

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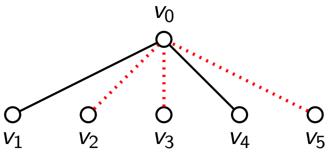

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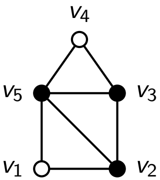

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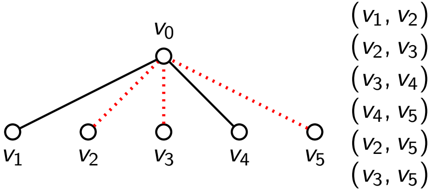

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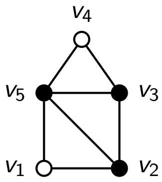

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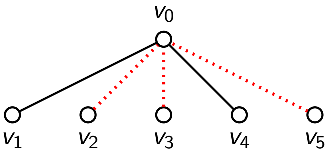

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More evidence that (subdivided) stars are important.

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Theorem (Kim et al., STOC'22)

The weighted versions of these problems are (randomized) FPT.

The proof uses directed flow augmentation.

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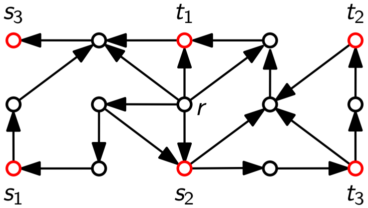

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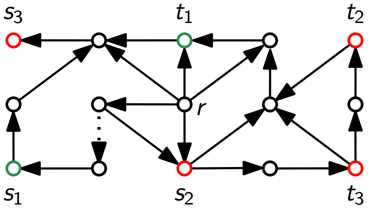

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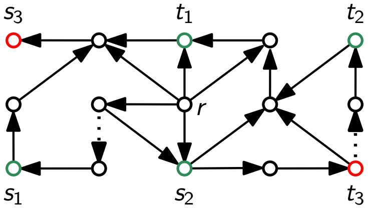

Weighted Digraph Pair-Cut

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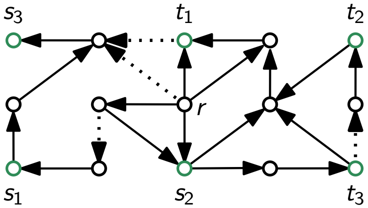

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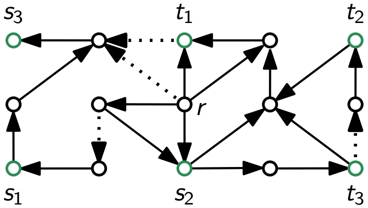
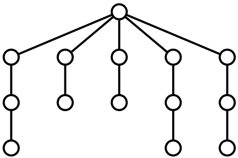


Recall: Weighted Digraph Pair-Cut is FPT.

Weighted Multicut on Subdivided Stars



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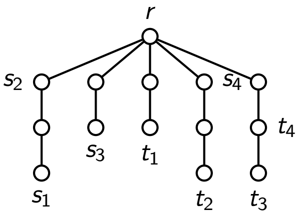

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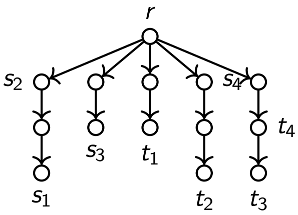

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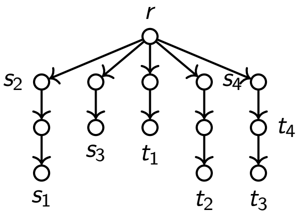

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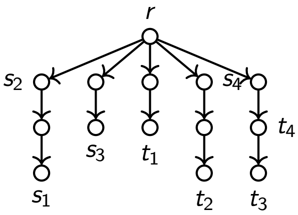

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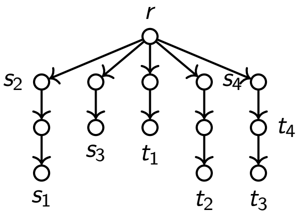

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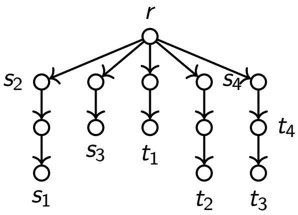

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Weighted Multicut on trees is FPT when parameterizing by the solution size k.

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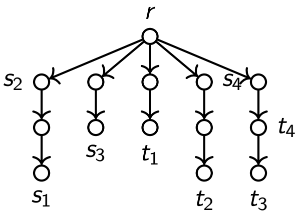

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Answers an implicit question by Bousquet et al. (STACS '09).

Edge Deletion vs. Vertex Deletion

Situation on trees:

	Multicut	
Unweighted	FPT: Simple branching algorithm	
Weighted		

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Our algorithm also works for the vertex version!

Conclusion

We use results for Weighted Digraph Pair-Cut to show the following:

Main Theorem 1

Weighted Multicut on trees can be solved in randomized time $2^{\mathcal{O}(k^4)} \cdot n^{\mathcal{O}(1)}$.

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Weighted Multicut without size constraint on trees with ℓ leaves can be solved in time $2^{\mathcal{O}(\ell^3)} \cdot n^{\mathcal{O}(1)}$.

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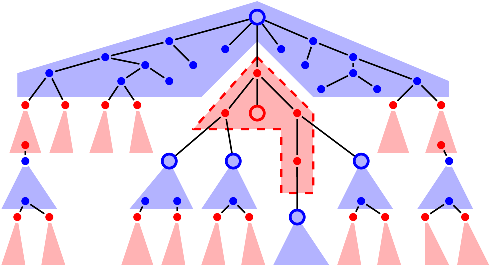
One more result generalizing Main Theorem 2 and a result by Guo and Niedermeier (2006) about request degree.

Full version: `arXiv:2205.10105`

Additional Material

(d, ℓ)-Light Instances

- Delete all vertices used for at most d terminal pair request.
- The closed neighborhood of the remaining components must has at most ℓ leaves.



Result for (d, ℓ)-Light Instances

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Parameter: request degree d and number of leaves ℓ

wMC on (d, ℓ)-light trees can be solved in time $3^d \cdot 2^{d\ell} \cdot 2^{\mathcal{O}(\ell^3)} \cdot n^{\mathcal{O}(1)}$
if we drop the size constraint.

Proof idea:

- For vertices with small ($\leq d$) request degree:
  Use dynamic programming.
- For components of vertices with large ($\geq d$) request degree:
  Use one of the new algorithms as subroutine as the component has at most ℓ leaves.

This implies a result by Guo and Niedermeier (2006) about the request degree d.

Parameterizing by Number of Leaves

Parameter: number of leaves ℓ

wMC on trees with ℓ leaves can be solved in time $2^{\mathcal{O}(\ell^3)} \cdot n^{\mathcal{O}(1)}$
if we drop the size constraint.

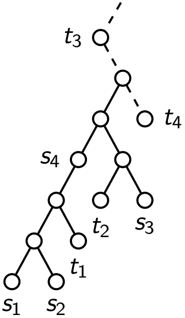
Proof idea:

- Use another result from Kim et al. '22 to solve the problem on paths and stars.
- Apply similar procedure as for previous algorithm to solve the problem on trees.

Algorithm – Main Idea

Preprocessing:

1 Compute a minimum *unweighted* solution X_{opt}.



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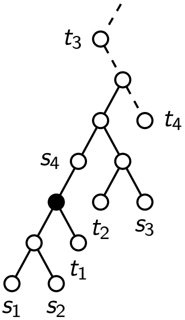

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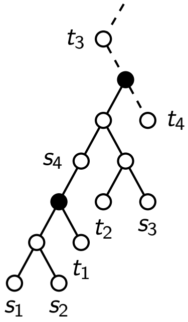

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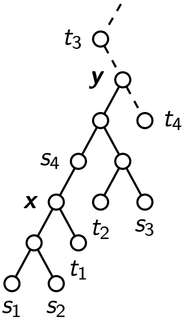

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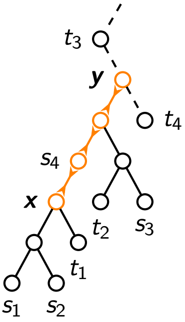

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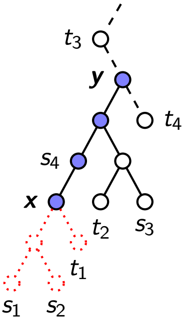
4. Case "there is such a vertex":
   For each vertex v between x and y:
   Update $\text{wt}(v) = \text{wt}(v) + \text{OPT}(T^{\dagger}_{v,x})$ (next step)
   Delete $T^{\dagger}_{x}$ and add the pair (x, y).
5. Recurse.



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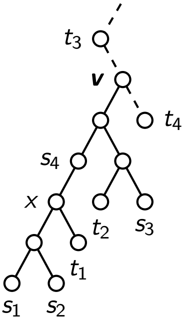

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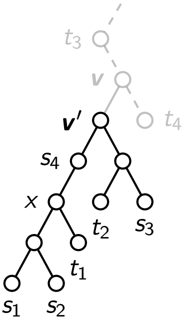

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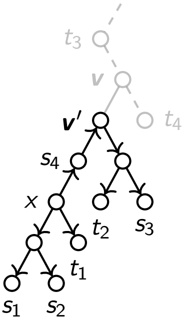

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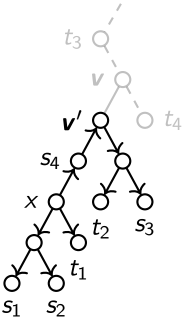

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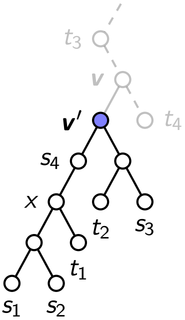

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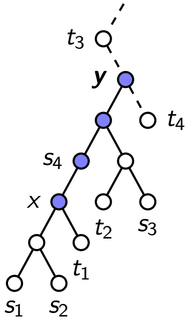

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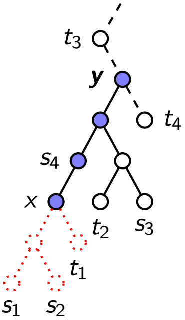

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Remove the subtree below x and the corresponding pairs.

Remove x from X and add (x, y) as a new pair to $\mathcal{P}$.

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$\implies$ Total running time is $k^{\mathcal{O}(k)} \cdot 2^{\mathcal{O}(k^4)} n^{\mathcal{O}(1)} = 2^{\mathcal{O}(k^4)} n^{\mathcal{O}(1)}$.

□