The Complexity of Contracting Bipartite Graphs into Small Cycles

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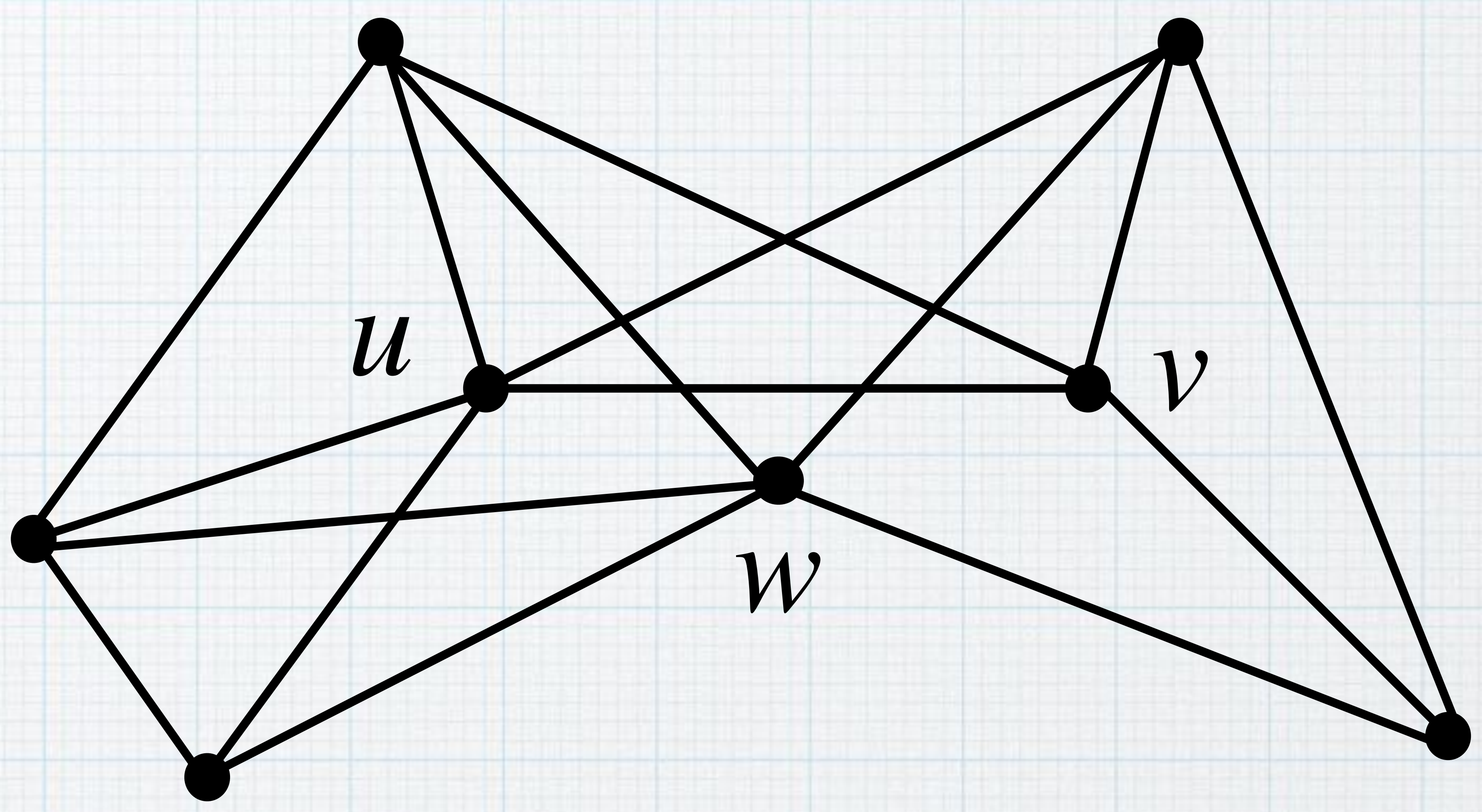
— Contracting edge uv in $E(G)$

- Delete vertices u and v.
- Add a new vertex w.
- Make it adjacent with all vertices that were adjacent with u or v.



G/F - graph obtained from G by contracting edges in F

$\mathscr{F}$-Edge Contraction

Given: Graph G, int k

Determine: $\exists$? subset F of $E(G)$ s.t. $|F| \leq k$ and $G/F \in \mathscr{F}$?

$\mathscr{F}$-Edge Contraction ← Vertex/Edge Addition or Deletion

Given: Graph G, int k

Determine: $\exists$? subset F of $E(G)$ s.t. $|F| \leq k$ and $G/F \in \mathscr{F}$?

Watanabe + T. N.('81): $\mathscr{F}$-Edge Contraction is NP-Complete if $\mathscr{F}$ satisfies certain properties.

Asano & Hirata ('83): $\mathscr{F}$-Edge Contraction is NP-Complete if $\mathscr{F}$ satisfies certain (even relaxed) properties.

Tree Edge Contraction

Cactus Edge Contraction

Planar Edge Contraction

Outer-planar Edge Contraction

Chordal Edge Contraction

Series-Parallel Edge Contraction

$\mathscr{F}$-Edge Contraction
Given: Graph G, int k
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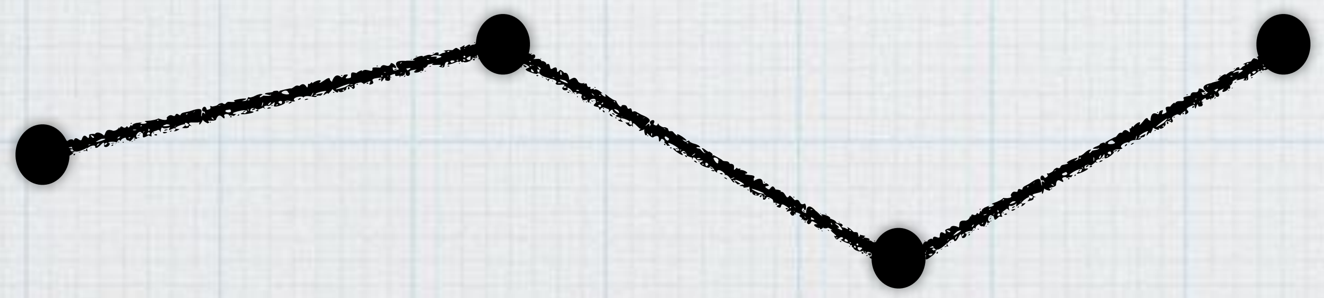
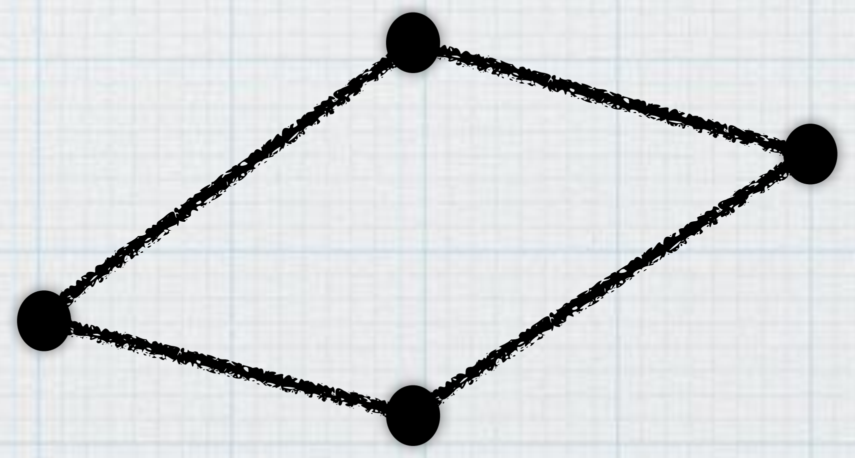
Brouwer & Veldman ('87): $\{P_4\}$-Edge Contraction is NP-Complete.
Also, $\{C_4\}$-Edge Contraction is NP-Complete.

P_4 

C_4 

$\mathscr{F}$-Edge Contraction

Given: Graph G, int k

Determine: $\exists$? subset F of $E(G)$ s.t. $|F| \leq k$ and $G/F \in \mathscr{F}$?

$\{P_4\}$-Edge Contraction

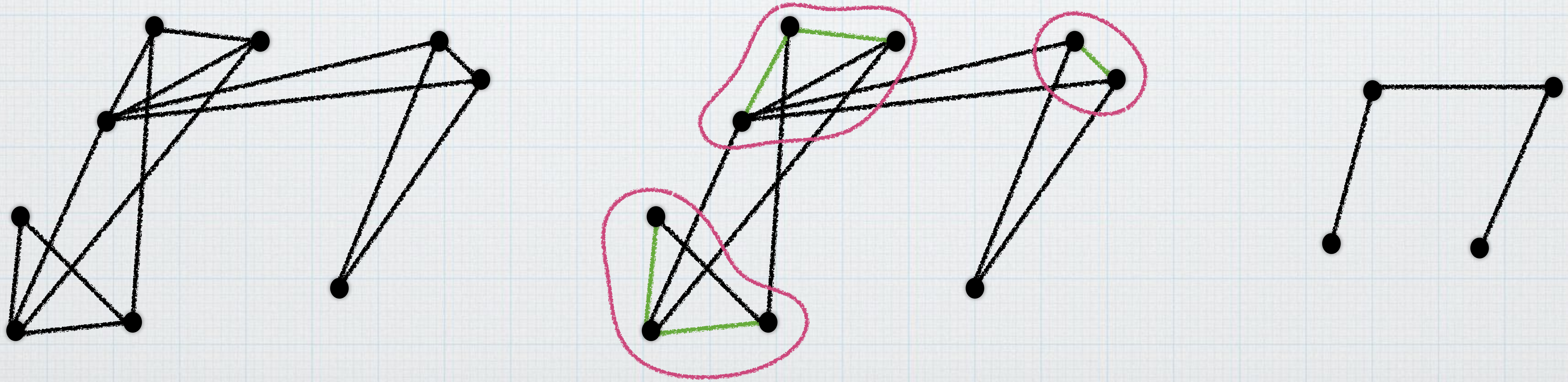
H-Contractibility

Given: Graph G

Determine: Can we obtain H from G by contracting edges?

P_4-Contractibility

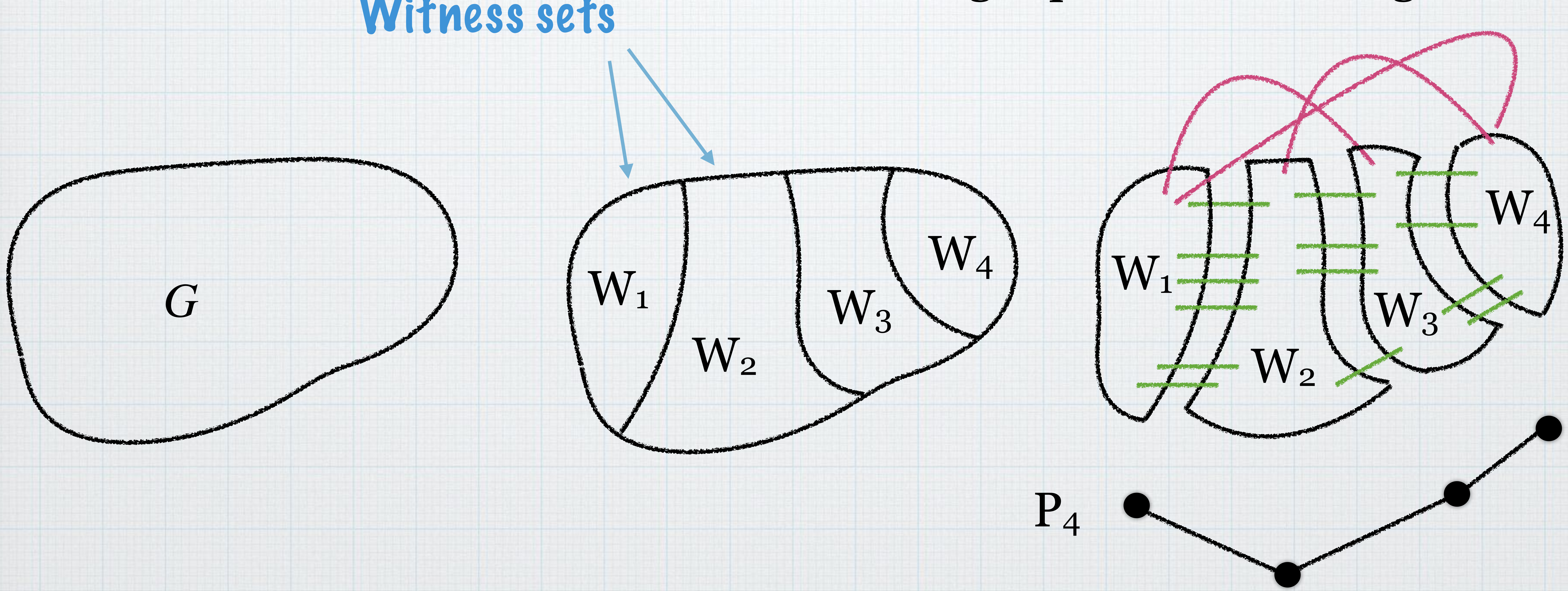
Vertex version, poly time.



G can be contracted to P_4 $\equiv$ $V(G)$ can be partitioned s.t.

Each W_i is connected

Green edges presents; red edges not



NP-Hardness for P_4-Contractibility

SAT:
Find an asst. of var s.t. for every clause,
- at least one of its literal is true.

$(x \vee \neg y \vee z)$

✓ (1,1,1)
✓ (1,0,1)

Not All Equal-SAT (NAE-SAT):
Find an asst. of var s.t. for every clause,
- at least one of its literal is true and
- not all of its literals are true.

$(x \vee \neg y \vee z)$

✓ (1,1,1)
✗ (1,0,1)

Positive Not All Equal-SAT (POS-NAE-SAT):
Every variable appears positively.
Find an asst. of var s.t. for every clause,
- at least one of its literal is true and
- not all of its literals are true.

$(x \vee w \vee z)$

✗ (1,1,1)
✗ (0,0,0)

Positive NAE-SAT

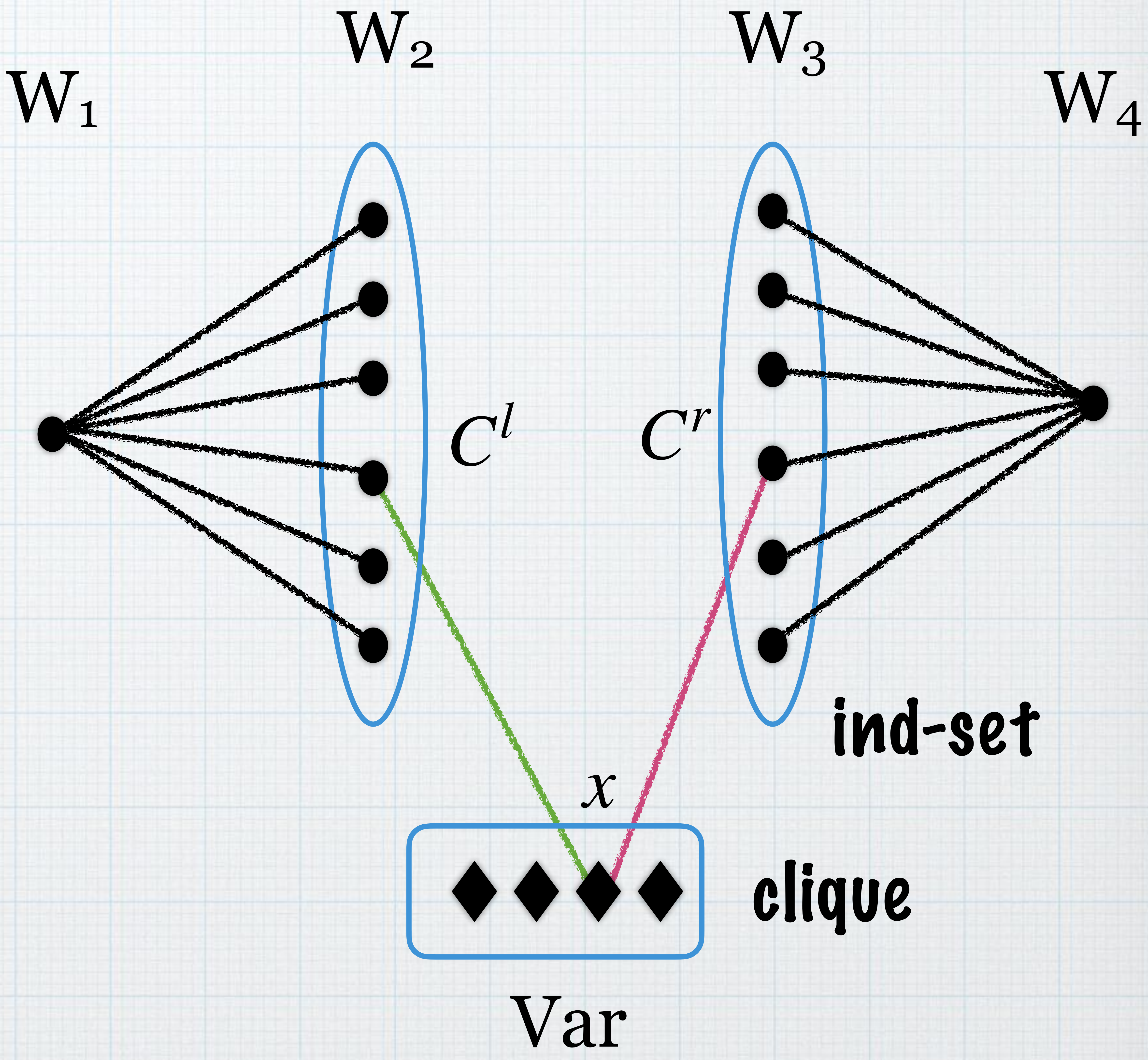
— Add two copies of clauses i.e. W_2, W_3 as independent sets

— Add two holders i.e. W_1, W_4

— Add vertices for variables and make them a clique

— Var x in clause C, add edges (x, C^l), (x, C^r)

P_4-Contractiblity



G can be contracted to P_4

V(G) can be partitioned s.t.

Each W_i is connected

Constraint on edges across partitions

Obj: Make both W_2, W_3 connected

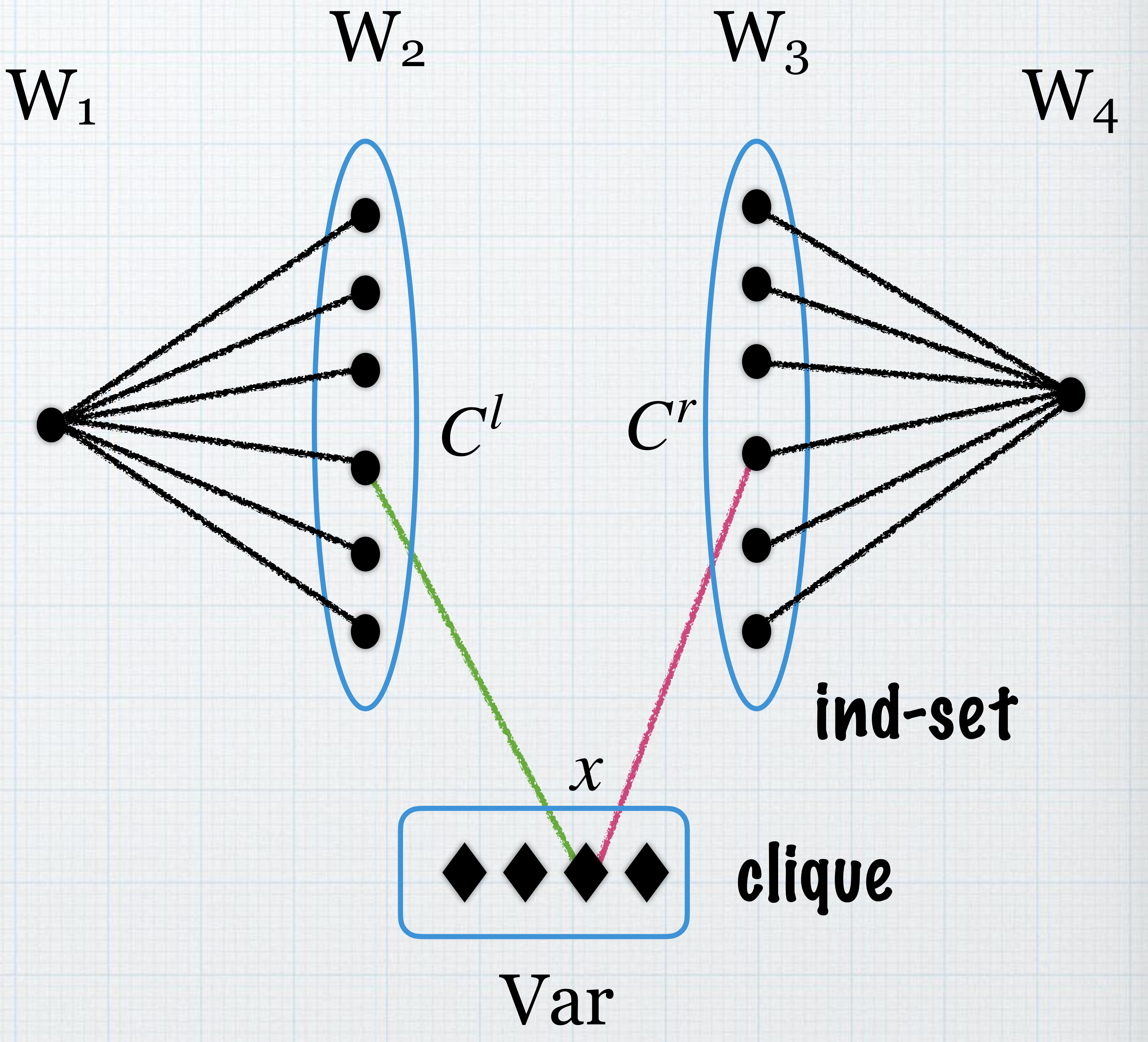
-W_2 collects all vars set to true

-W_3 collects all vars set to false

W_2, W_3 contain copies of all clauses.

$\Rightarrow$ for every clause,

- at least one var is set to true
- not all vars are set to true



H-Contractibility
Given: Graph G
Determine: Can we obtain H from G by contracting edges?

Brouwer & Veldman ('87): P_4-Contractiblity, C_4-Contractiblity are NP-Complete.

Levin, Paulusma, and Woeginger (WG '03):

P vs NP dichotomy of H-Contractiblity when $|V(H)| \leq 5$.

Far from understanding H-Contractiblity on general graphs.

H-Contractiblity on special graph classes

Chordal	Planar	Bipartite	Claw-free

Closed under edge contractions

Not closed under edge contractions

	Gen. Graph		Chordal		Planar		Bipartite		Claw-free	
	Path	Cycle	Path	Cycle	Path	Cycle	Path	Cycle	Path	Cycle
3	P	P	P	P	P	P	P	P	P	P
4	NP	NP					?	?	P	?
5							?	?	?	?
6							NP	NP	?	NP
7					P			?	NP	
ℓ			P	P	XP	P		?		

Our Results

Thm: C_5-Contractiblity on bipartite graph is NP-Complete.

Thm: C_4-Contractiblity on bipartite graph is NP-Complete.

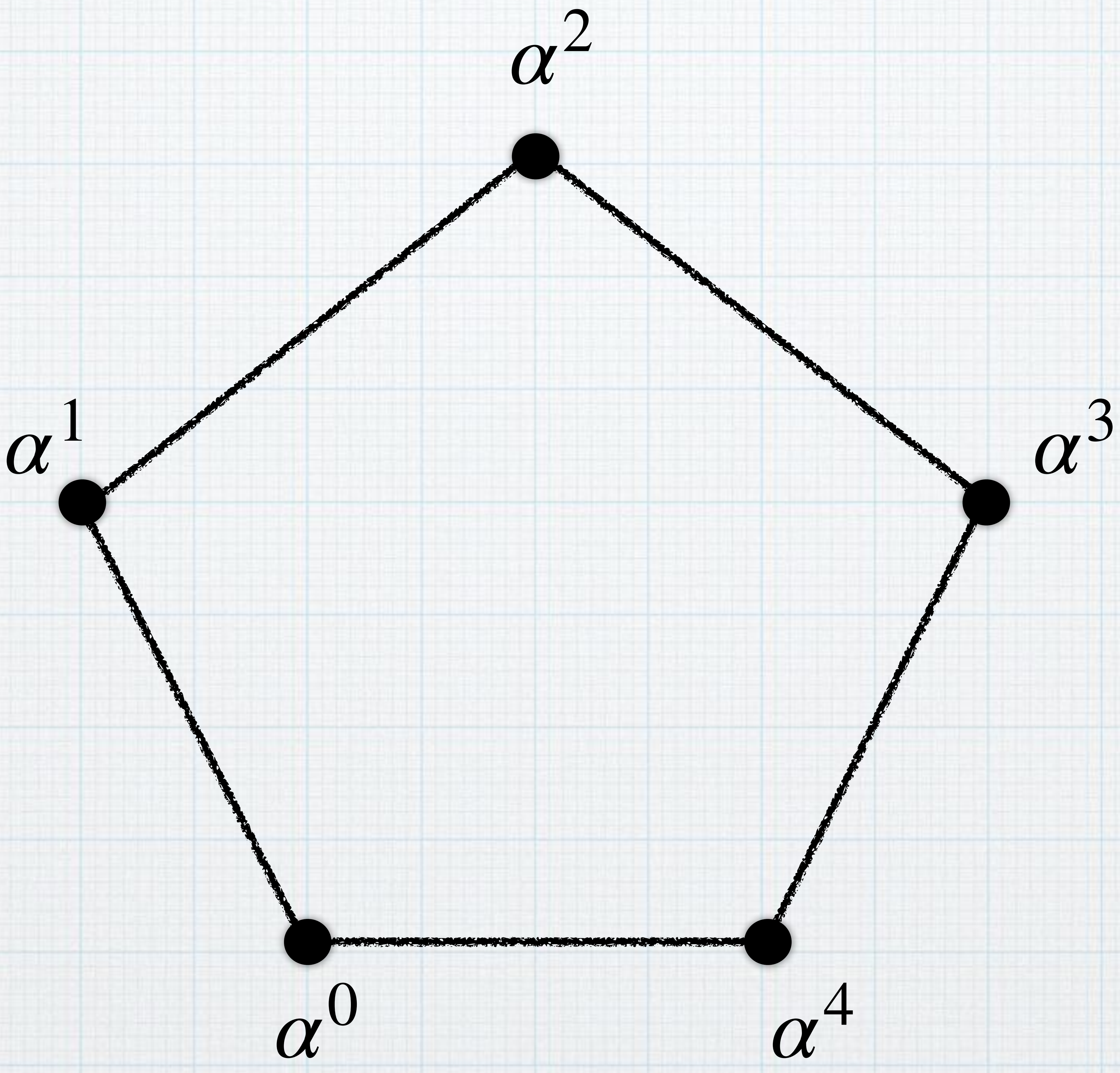
Thm: C_4-Contractiblity on K_4-free graphs of diameter 2 is NP-Complete.

In the journal version

Thm: C_ℓ-Contractibility on Bipartite Graph is NP-Complete
for every $\ell \geq 6$.

Positive NAE-SAT

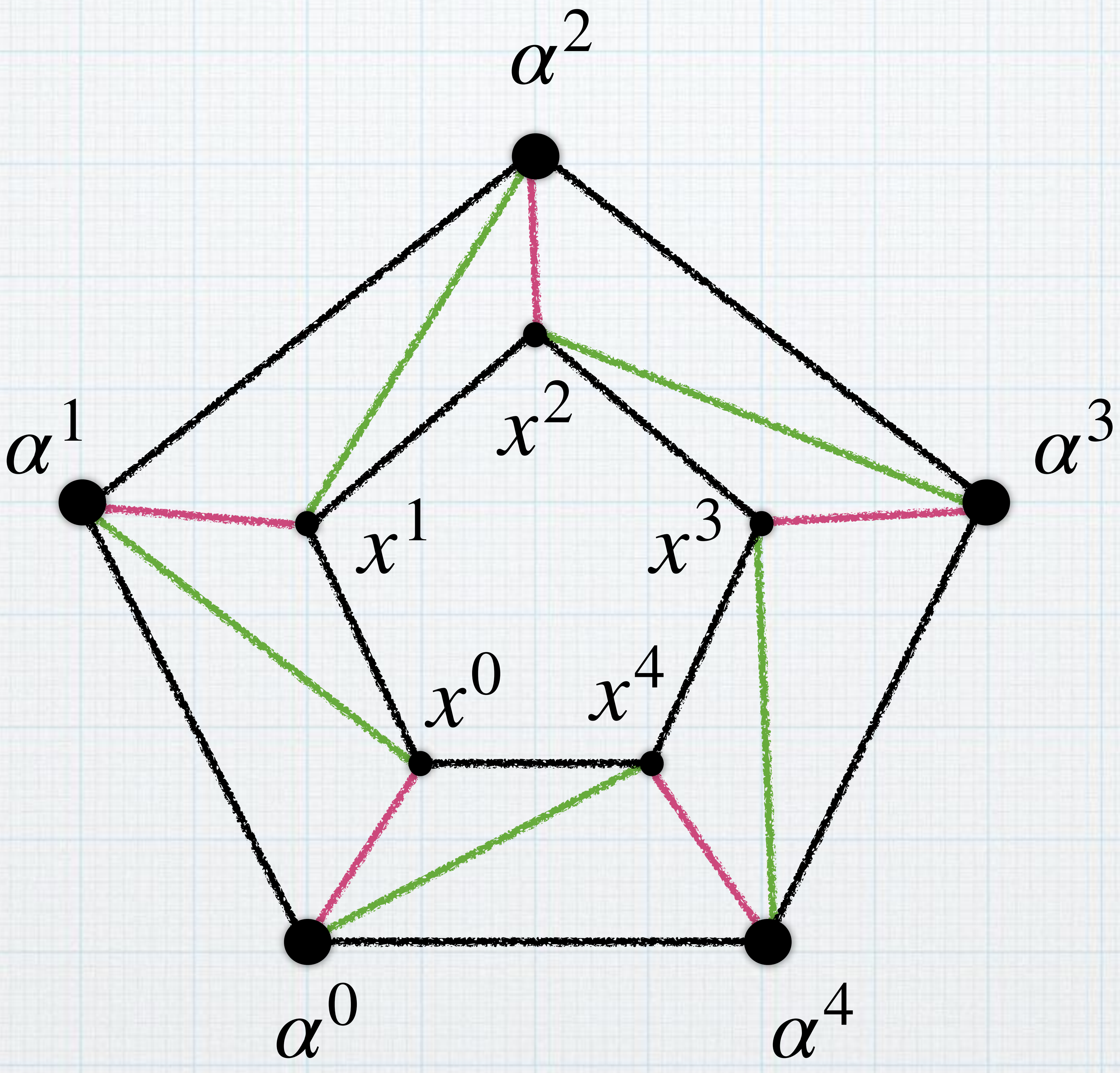
C_5-Contractiblity



1. Add a base cycle i.e. no edge in this gets contracted.
(Most technical part)

Positive NAE-SAT

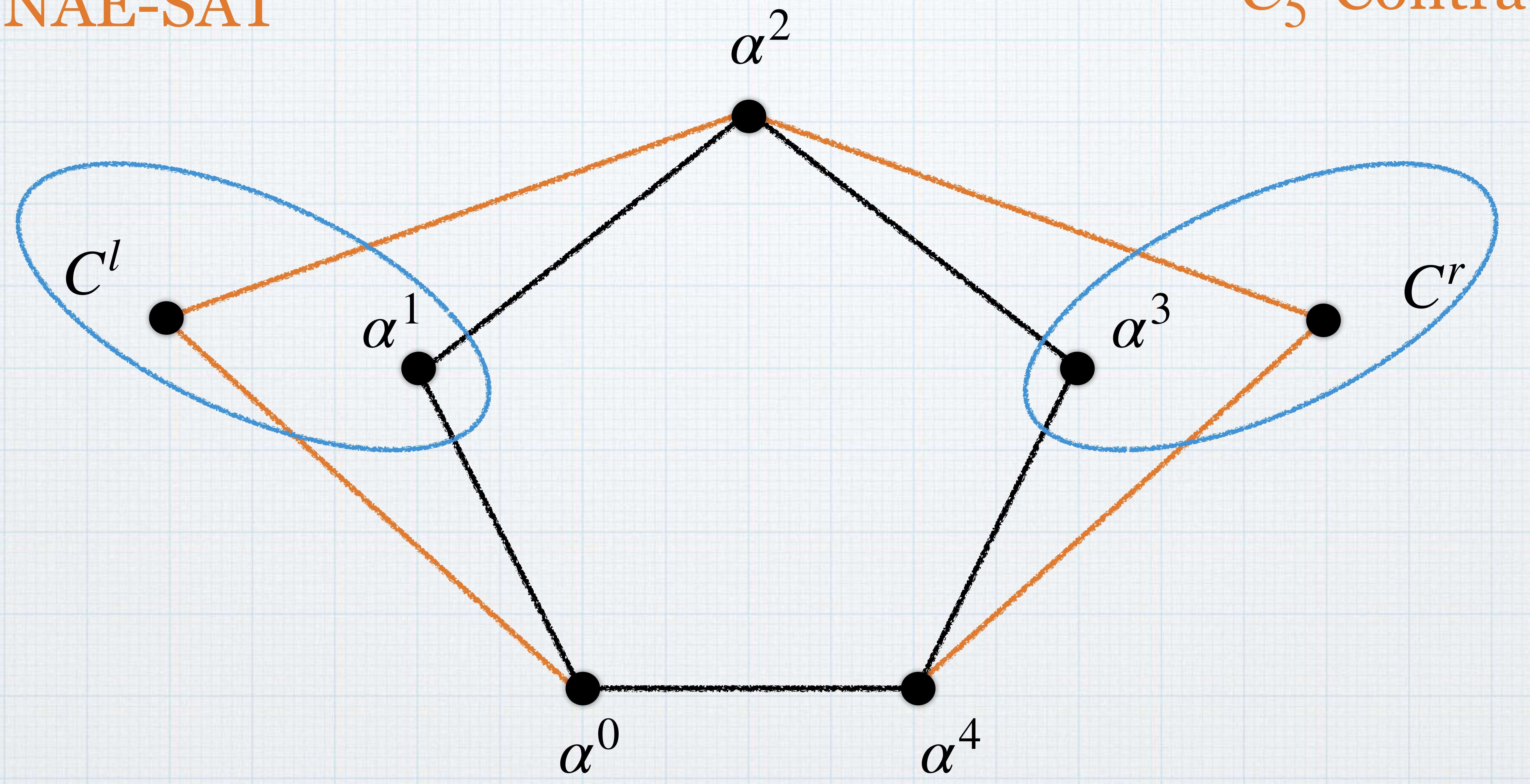
C_5-Contractiblity



2. Add a concentric cycle for each variable.
Green edges contracted = variable set to positive
Red edges contracted = variable set to negative

Positive NAE-SAT

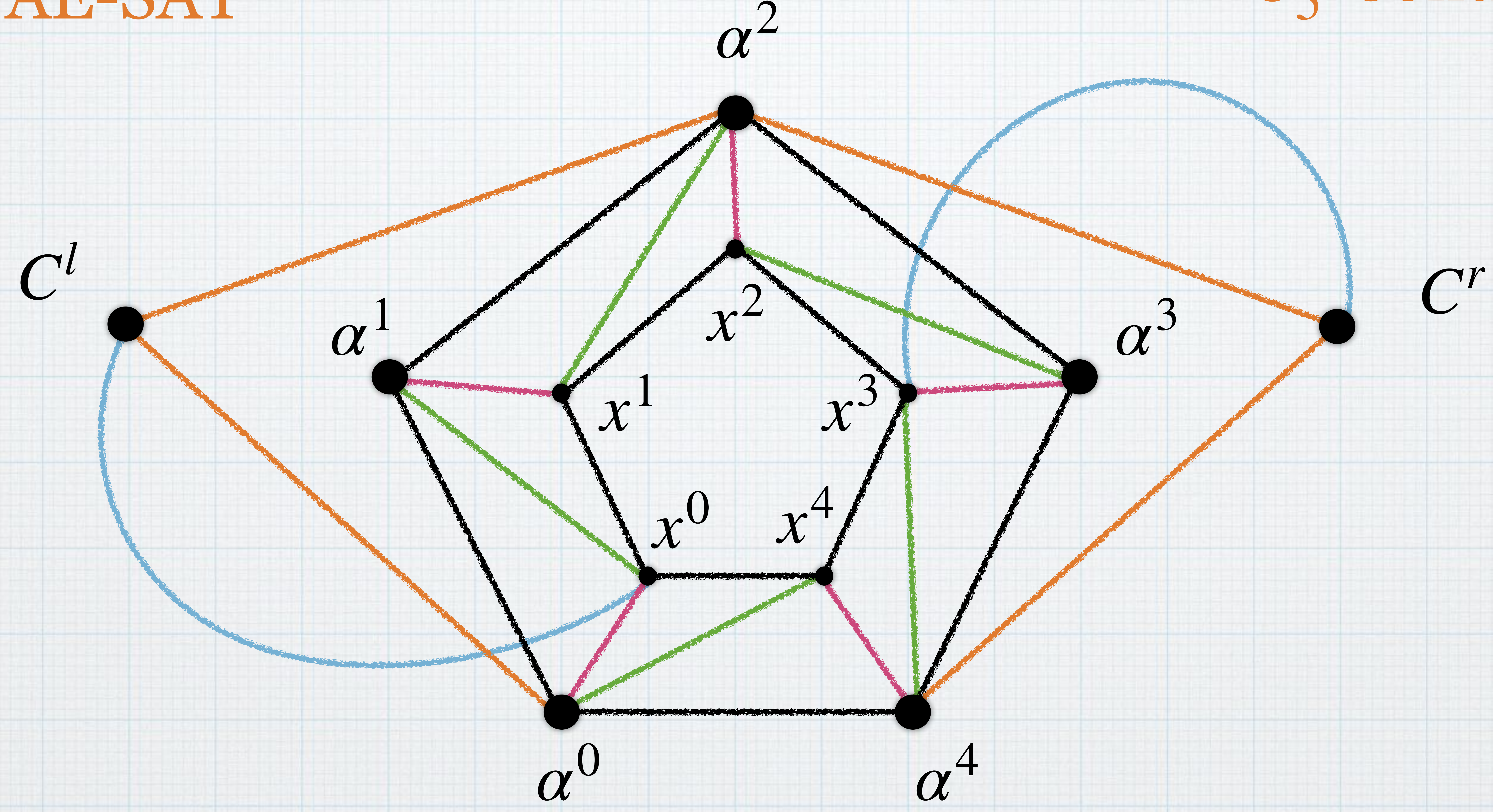
C_5-Contractiblity



3. Add a two vertices for each clause. Anchor them.
C^l responsible for setting variable to true
C^r responsible for setting variable to false.

Positive NAE-SAT

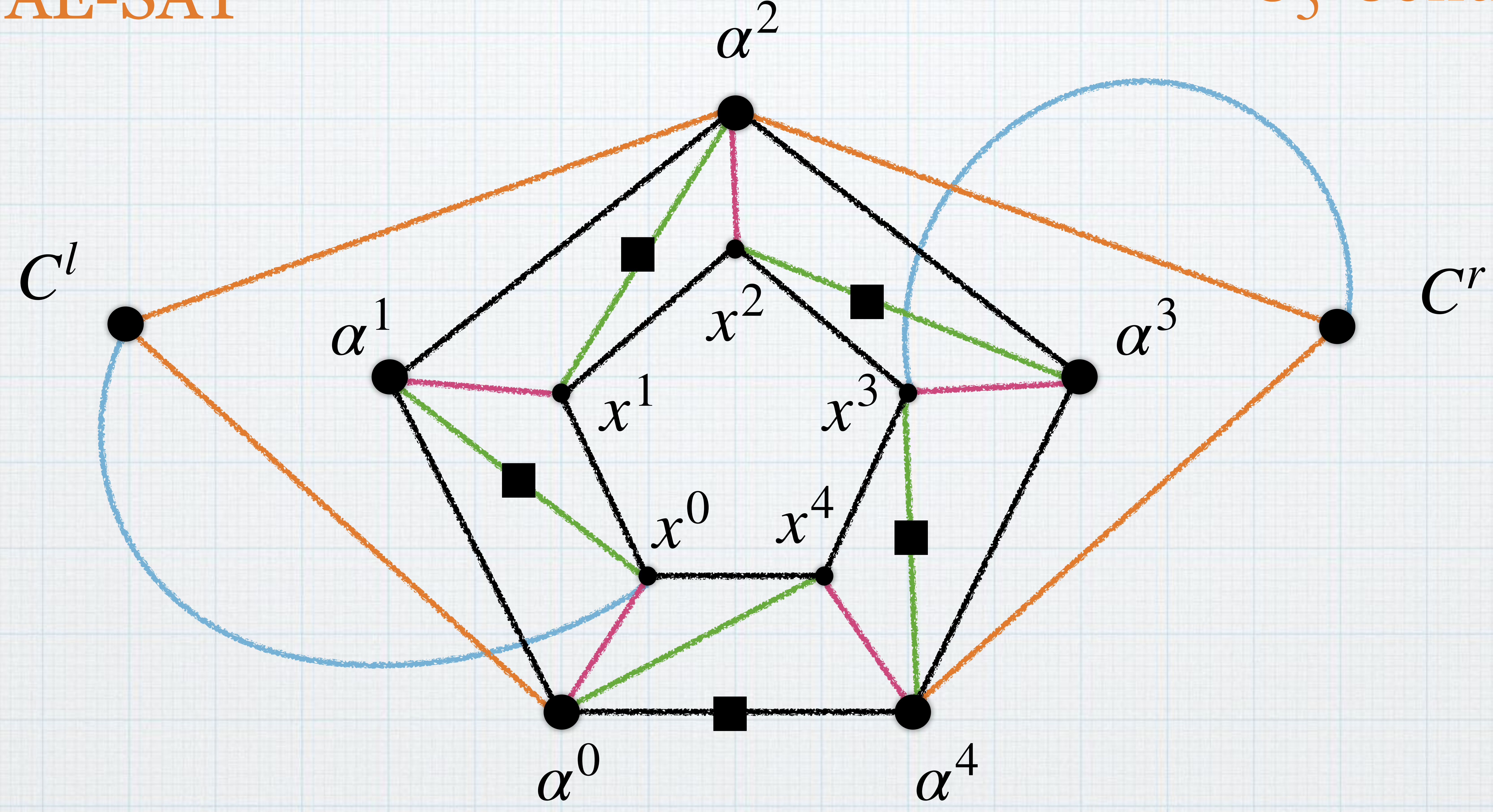
C_5-Contractiblity



4. Add variable-clause connection if $x \in C$.
Variable x can help either C^l or C^r.

Positive NAE-SAT

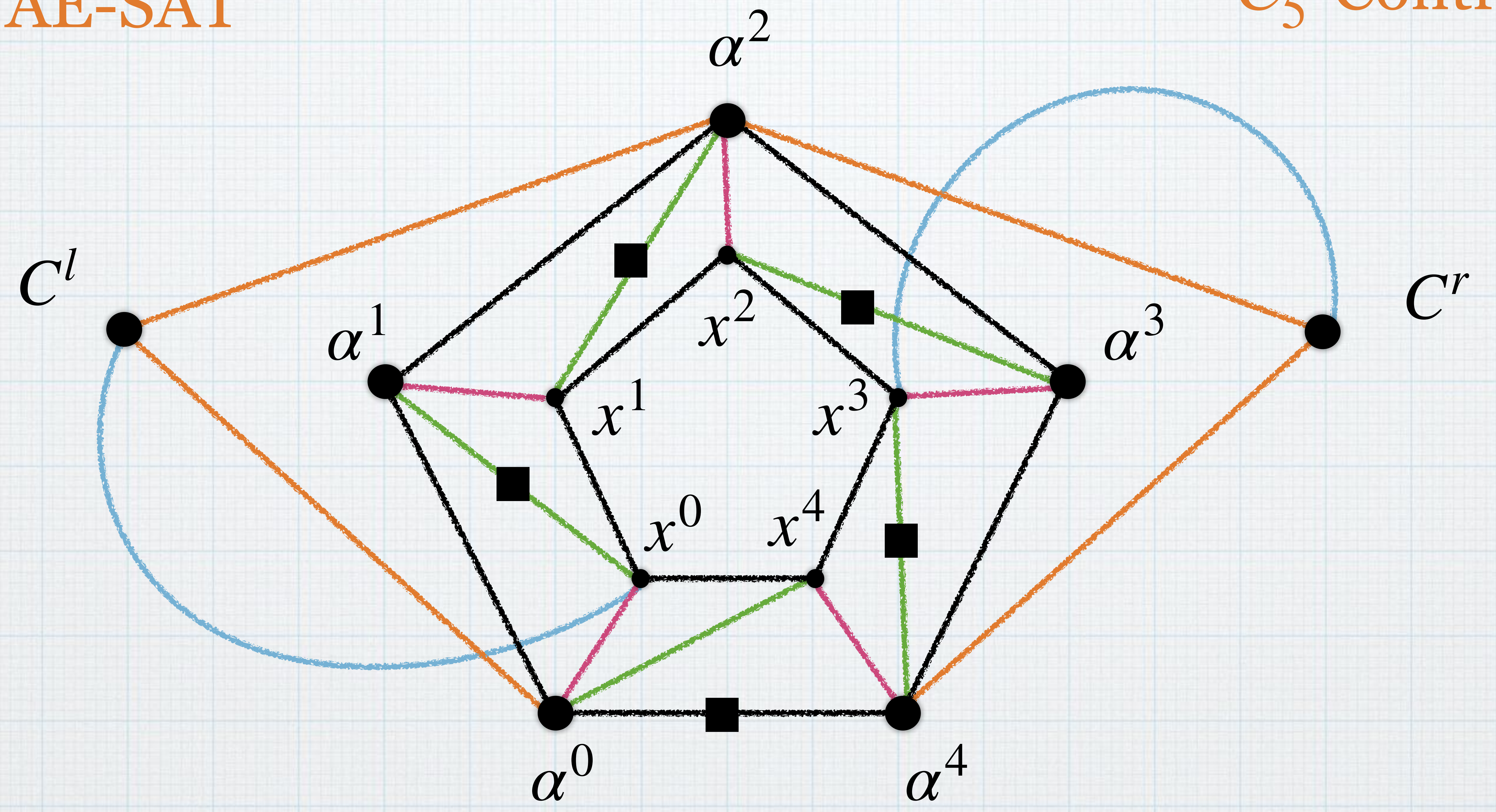
C_5-Contractiblity



5. For C_5-Contractiblity, we can subdivide some edges safely.

Positive NAE-SAT

C_5-Contractiblity



Thm: C_5-Contractiblity on bipartite graph is NP-Complete.

	Gen. Graph		Chordal		Planar		Bipartite		Claw-free	
	Path	Cycle	Path	Cycle	Path	Cycle	Path	Cycle	Path	Cycle
3	P	P	P	P	P	P	P	P	P	P
4	NP	NP					?	NP	P	?
5							?	NP	?	?
6							NP	NP	?	NP
7					P			NP	NP	
ℓ			P	P	XP	P		NP		

Thank you!