Parameterized Complexity of Biclique Contraction and Balanced Biclique Contraction

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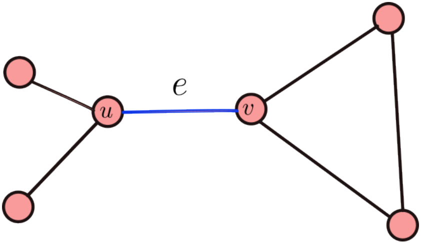
Edge Contractions



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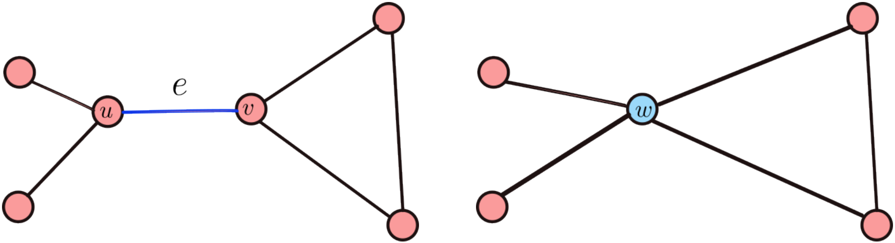
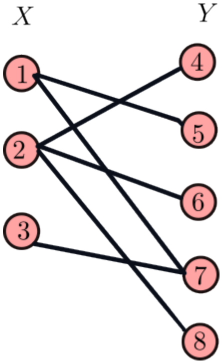


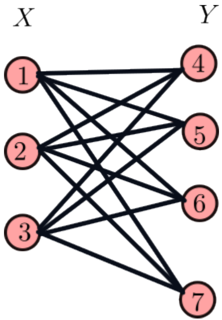
Bipartite Graph

- A graph is bipartite if its vertex set has a bipartition $\langle X, Y \rangle$, such that X and Y are independent sets.
- A set of vertices X is an *independent set*: if no two vertices in X are adjacent.



Biclique and Balanced Biclique

- A bipartite graph G with bipartition $\langle X, Y \rangle$ is a *biclique* if every vertex in X is adjacent to every vertex in Y.



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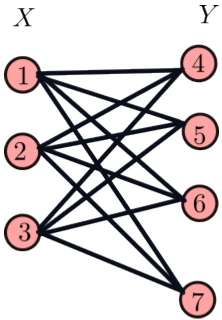
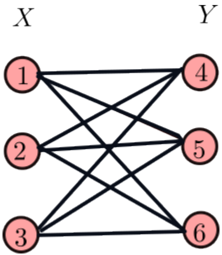


- A biclique G with bipartition $\langle X, Y\rangle$ is a *balanced biclique* if $|X| = |Y|$.



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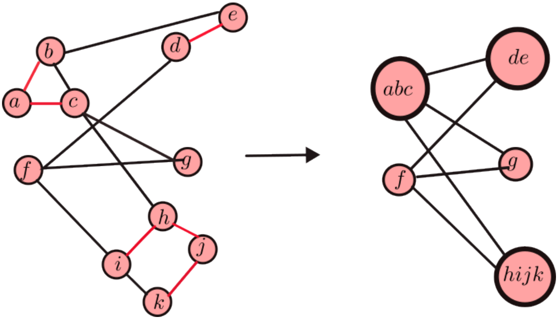
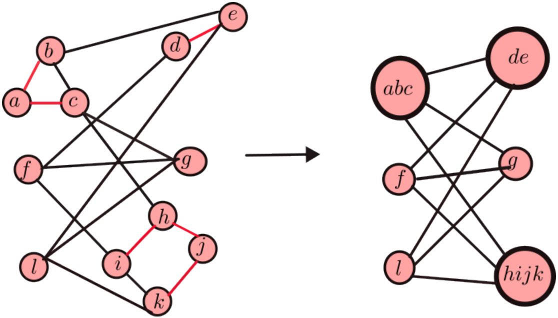
Biclique Contraction

Biclique Contraction
Input: A graph G on n vertices and an integer k.
Question: Can we contract $\leq k$ edges in G to obtain a biclique?
Parameter: k.



Balanced Biclique Contraction
Input: A graph G on n vertices and an integer k.
Question: Can we contract $\leq k$ edges in G to obtain a balanced biclique?
Parameter: k.



Our Results

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- Biclique Contraction is FPT via an $\mathcal{O}^*(25.904^k)$-time algorithm.

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- Balanced Biclique Contraction admits a quadratic vertex kernel
- Biclique Contraction does not admit any polynomial kernel unless NP $\subseteq$ coNP/poly.

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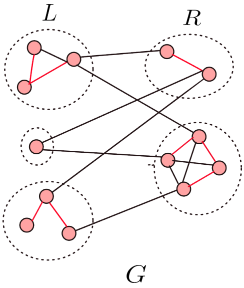
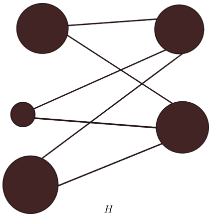
Contracting Graphs to *Bicliques*

k-constrained valid partition

(G, k) is a YES-instance of BICLIQUE CONTRACTION iff $V(G)$ can be partioned into two parts $\langle L, R \rangle$:

1. Size_of(Span_forest of $G[L]$) + Size_of(Span_forest of $G[R]$) $\leq k$.
2. Every component of $G[L]$ is adjacent to every component of $G[R]$.





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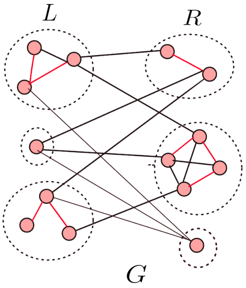
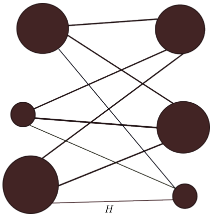
k-constrained valid balanced partition

(G, k) is a YES-instance of BALANCED BICLIQUE CONTRACTION iff $V(G)$ can be partioned into two parts $\langle L, R\rangle$:

1. $\langle L, R\rangle$ is a k-constrained valid partition.
2. Number of components of $G[L]$ = Number of components of $G[R]$.





NP-completeness of Biclique Contraction

- Red-Blue Dominating Set $\leq_P$ Biclique Contraction.

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NP-completeness of BICLIQUE CONTRACTION

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  - $(G, R, B, \kappa) \longmapsto (H, k)$ where H is bipartite, $k = |B| + \kappa$

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NP-completeness of BICLIQUE CONTRACTION

- RED-BLUE DOMINATING SET $\leq_P$ BICLIQUE CONTRACTION.
  - $(G, R, B, \kappa) \longmapsto (H, k)$ where H is bipartite, $k = |B| + \kappa$
- BICLIQUE CONTRACTION is NP-complete even when restricted to bipartite graphs.
- BICLIQUE CONTRACTION does not admit any polynomial compression (or kernel)

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NP-completeness of Balanced Biclique Contraction

- Hypergraph 2-Coloring $\leq_P$ Bal Biclique Contraction.

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NP-completeness of Balanced Biclique Contraction

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  - $(\mathcal{G}) \longmapsto (G, \kappa)$, κ is a function of the number of vertices and the number of hyperedges in $\mathcal{G}$
- Bal Biclique Contraction is NP-complete even when restricted to bipartite graphs.

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FPT Algorithm

BICLIQUE CONTRACTION and BALANCED BICLIQUE CONTRACTION can be solved in $\mathcal{O}^*(25.904^k)$ time.

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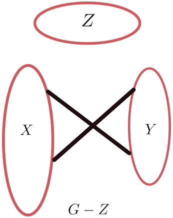
Biclique modulator

Observation 1

If G is k-contractible to a (balanced) biclique, then G has a biclique modulator of size at most $2k$.



Proposition 1 [Hüffner et al., 2010]

A Biclique modulator Z of size at most k can be obtained in $\mathcal{O}^*(1.4^k)$ time.

FPT Algorithm for Biclique Contraction

- Let (G, k) be an instance of Biclique Contraction.

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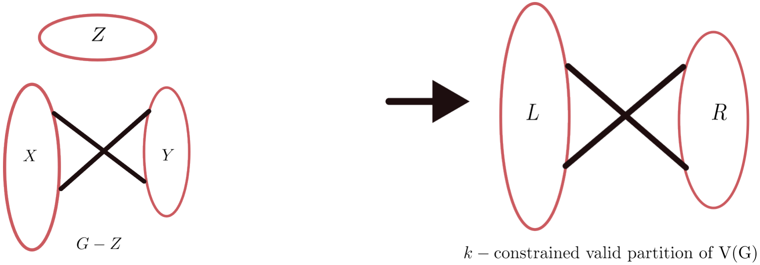
- If G does not has a biclique modulator Z of size at most $2k$:
  -Declare No

FPT Algorithm for BICLIQUE CONTRACTION

- Let (G, k) be an instance of BICLIQUE CONTRACTION.
- If G does not has a biclique modulator Z of size at most $2k$:
  -Declare NO
- Otherwise:



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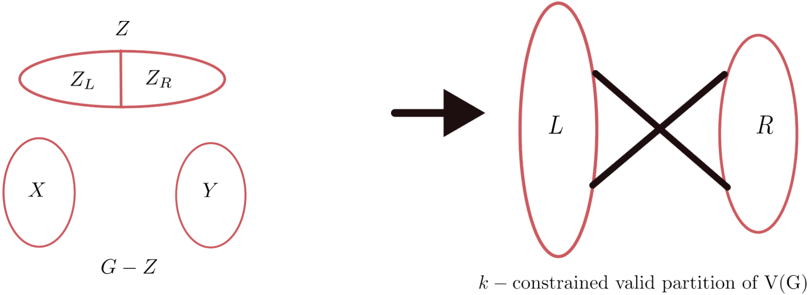
FPT Algorithm for Biclique Contraction

Case 1: $X \cup Y = \emptyset$

Guess the partition of Z in $\mathcal{O}^*(2^{|Z|})$ time.



FPT Algorithm for Biclique Contraction

Case 2: $X \cup Y \neq \emptyset$ where $X = \emptyset$

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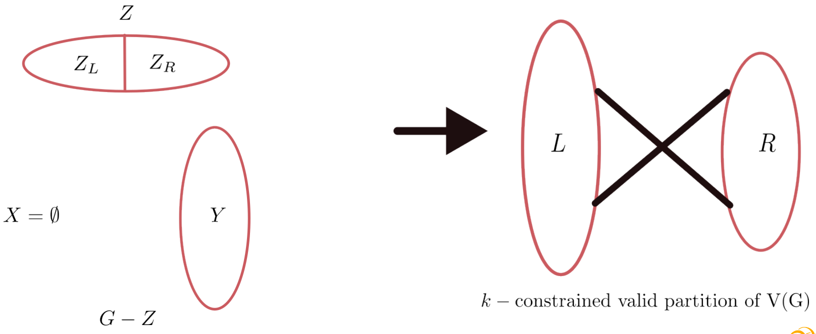
Case 2.1: $Y \cap L \neq \emptyset$ or $Y \cap R \neq \emptyset$ but not both.

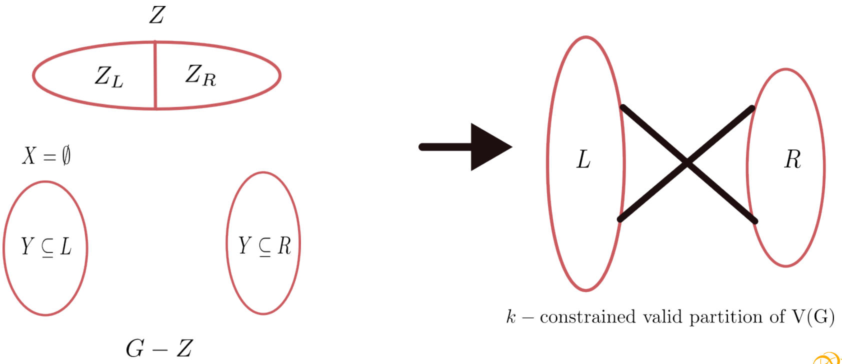
Determine if $\langle Z_L, Z_R \cup Y \rangle$ is a k-constrained valid partition.



The total running time - $\mathcal{O}^*(2^{|Z|})$

Case 2: $X \cup Y \neq \emptyset$ where $X = \emptyset$

Case 2.2: $Y \cap L \neq \emptyset$ and $Y \cap R \neq \emptyset$.

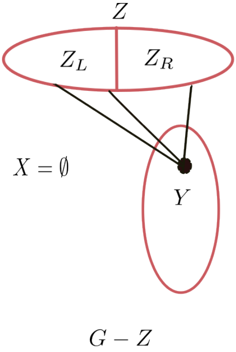


FPT Algorithm for Biclique Contraction

Branching Rule 1 for Case 2.2

If there is a vertex $v \in Y$ such that $N(v) \cap Z_L \neq \emptyset$, $N(v) \cap Z_R \neq \emptyset$ and $|N(v) \cap (Z_L \cup Z_R)| > 2$, then branch into the following.

- Contract all edges in $E(v, Z_L)$ and dec k.
- Contract all edges in $E(v, Z_R)$ and dec k.



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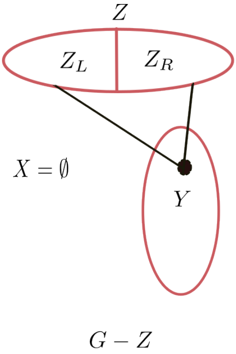
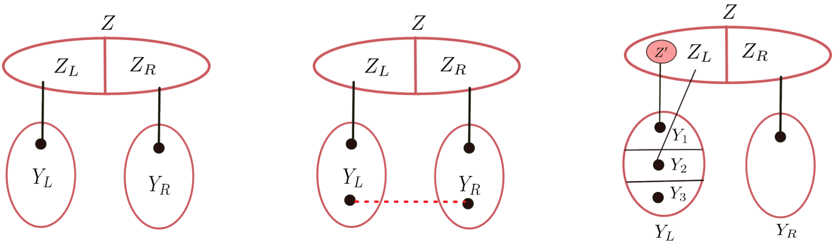
Preprocessing Rule 1 for Case 2.2

If there is a vertex $v \in Y$ of degree 2 such that $N(v) \cap Z_L \neq \emptyset$ and $N(v) \cap Z_R \neq \emptyset$, then contract edges in $E(v, Z_L)$ and dec k.



When neither Branching Rule 1 nor Prepossessing Rule 1 is applicable.



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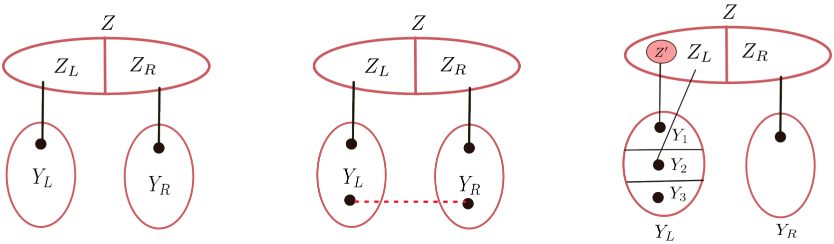

- Check if $(Z_L \cup Y_L, Z_R \cup Y_R)$ is a k-constrained valid partition.

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FPT Algorithm for BICLIQUE CONTRACTION

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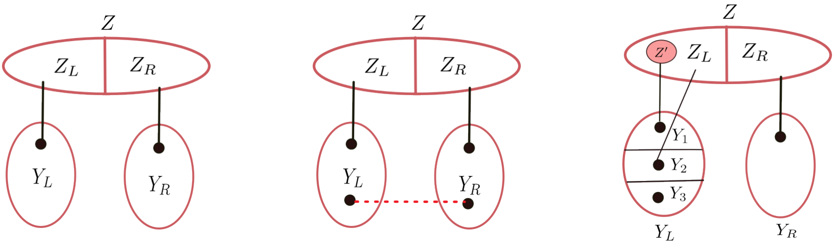

- Check if $(Z_L \cup Y_L, Z_R \cup Y_R)$ is a k-constrained valid partition.
- Contract all edges in $E(Y_R, Z_R)$ and dec k.

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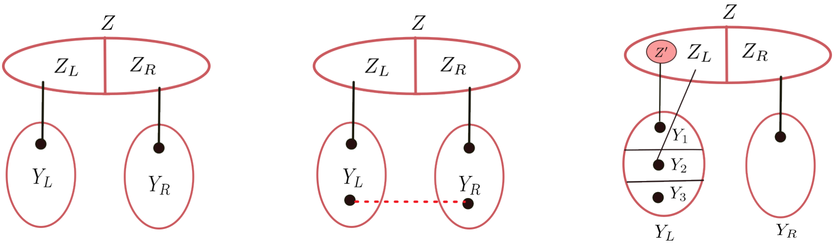

- Check if $(Z_L \cup Y_L, Z_R \cup Y_R)$ is a k-constrained valid partition.
- Contract all edges in $E(Y_R, Z_R)$ and dec k.
- Check if $(Z_L \cup Y_1 \cup Y_2, Z_R \cup Y_3)$ is a k-constrained valid partition.

Total running time - $\mathcal{O}^*(1.619^k 2^{2|Z|})$

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FPT Algorithm for BICLIQUE CONTRACTION

- **Case 1:** $X \cup Y = \emptyset$ - $\mathcal{O}^*(2^{|Z|})$
- **Case 2:** $X \cap L = \emptyset$ and $X \cap R = \emptyset$:
  1. Either $Y \cap L \neq \emptyset$ or $Y \cap R \neq \emptyset$ but not both - $\mathcal{O}^*(2^{|Z|})$
  2. $Y \cap L \neq \emptyset$ and $Y \cap R \neq \emptyset$. Similar Case - $\mathcal{O}^*(1.619^k 2^{2|Z|})$
- **Case 3:** Either $X \cap L \neq \emptyset$ or $X \cap R \neq \emptyset$ but not both:
  1. Either $Y \cap L \neq \emptyset$ or $Y \cap R \neq \emptyset$ but not both - $\mathcal{O}^*(2^{|Z|})$
  2. $Y \cap L \neq \emptyset$ and $Y \cap R \neq \emptyset$. Similar Case - $\mathcal{O}^*(1.619^k 2^{2|Z|})$
- **Case 4:** $X \cap L \neq \emptyset$ and $X \cap R \neq \emptyset$:
  1. Either $Y \cap L \neq \emptyset$ or $Y \cap R \neq \emptyset$ but not both. Similar Case - $\mathcal{O}^*(1.619^k 2^{2|Z|})$
  2. $Y \cap L \neq \emptyset$ and $Y \cap R \neq \emptyset$ - $\mathcal{O}^*(2^{|Z|+k})$

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FPT Algorithm for Biclique Contraction

- **Case 1:** $X \cup Y = \emptyset$ - $\mathcal{O}^*(2^{|Z|})$
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  2. $Y \cap L \neq \emptyset$ and $Y \cap R \neq \emptyset$ - $\mathcal{O}^*(2^{|Z|+k})$

Biclique Contraction can be solved in time $\mathcal{O}^*(25.904^k)$

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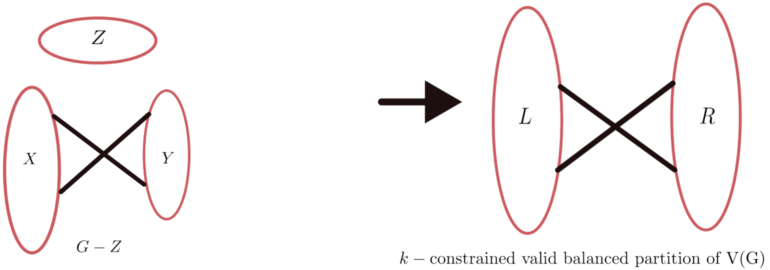
- Balanced Biclique Contraction: to determine if the partition is a k-constrained valid balanced partition or not.



FPT Algorithm for Bal Biclique Contraction

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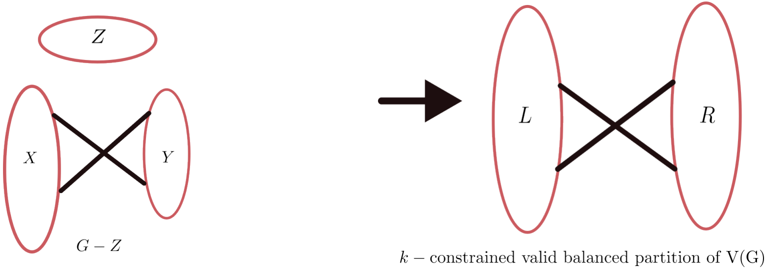

Balanced Biclique Contraction can be solved in time $\mathcal{O}^*(25.904^k)$

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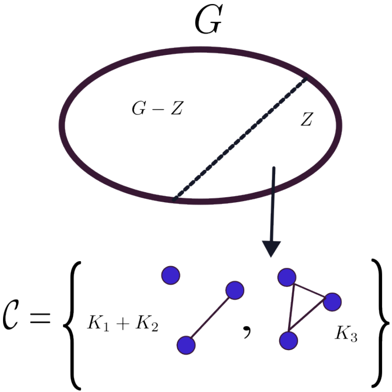
Kernelization of Balanced Biclique Contraction

- Admits **Quadratic Vertex Kernel** for Balanced Biclique Contraction using a sequence of reduction rules.

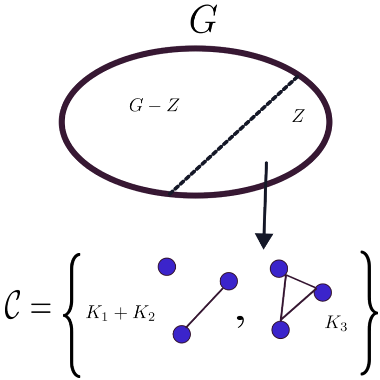
Polynomial Kernel for Bal Biclique Contraction

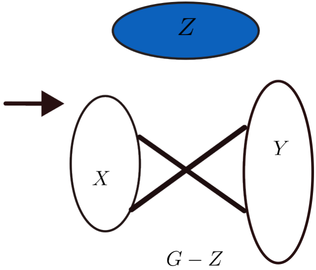
$\mathcal{C}$ be a maximal collection of vertex-disjoint $K_1 + K_2$ and K_3 in G and $Z = \bigcup_{C \in \mathcal{C}} V(C)$.



Reduction Rule 1

If $|Z| > 6k$, then return a trivial No-instance.



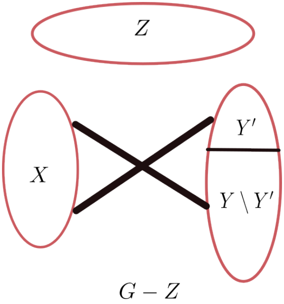


Subsequently, we assume that $|Z| \leq 6k$, $|X| \leq |Y|$, $|Y| \geq k + 3$

Reduction Rule 2

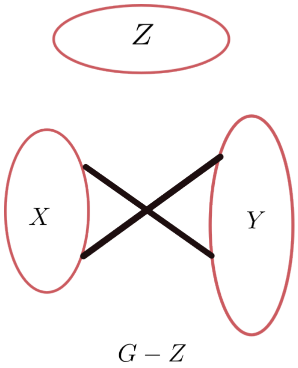
Subsequently, we assume that $|Z| \leq 6k$, $|X| \leq |Y|$, $|Y| \geq k+3$

If $|Y| > |X| + |Z| + k$, then return a trivial NO-instance.



Observation

If (G, k) is a $\textsc{Yes}$-instance then, in any k-constrained valid balanced partition $\langle L, R \rangle$ of $V(G)$ either $X \subseteq L$, $Y \subseteq R$ or $X \subseteq R$, $Y \subseteq L$.



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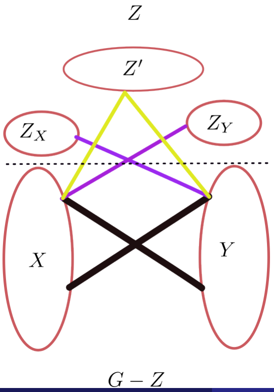
Reduction Rule 3

If there is an edge in $E(X, Z_X) \cup E(Y, Z_Y) \cup E(Z_X) \cup E(Z_Y)$, then contract it and decrease k by 1.

Z_X has more than $k+1$ neighbours in Y
Z_Y has more than $k+1$ neighbours in X



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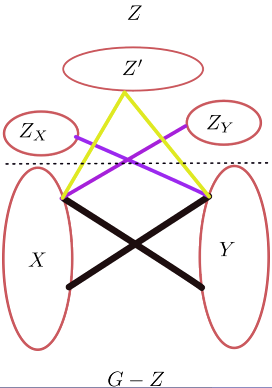

- if $Z_X \cap Z_Y \neq \emptyset$, then (G, k) is a No-instance.

Reduction Rule 3

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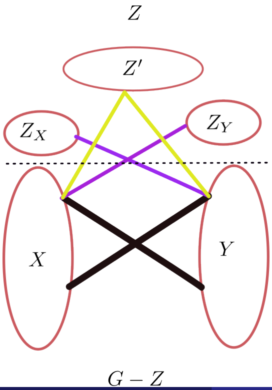
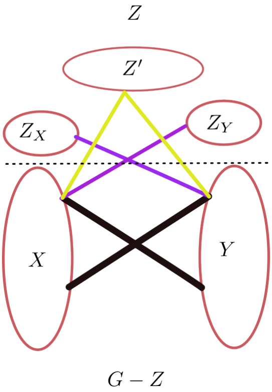


- if $Z_X \cap Z_Y \neq \emptyset$, then (G, k) is a No-instance.
- Z', Z_X and Z_Y partition Z.

- Mark all vertices in Z



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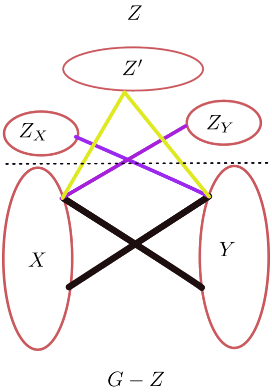
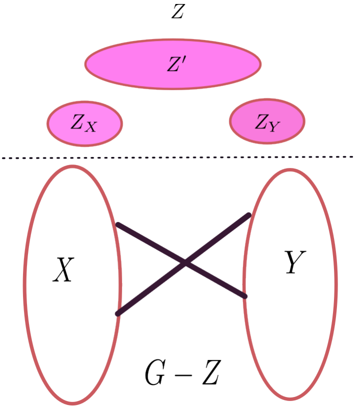
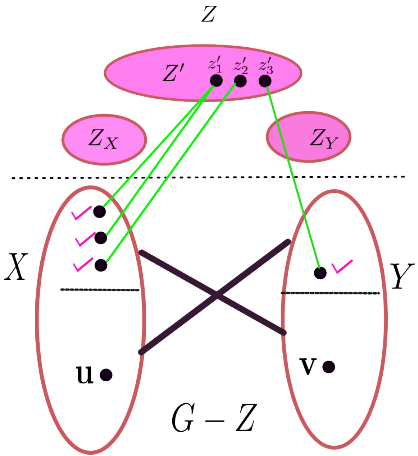


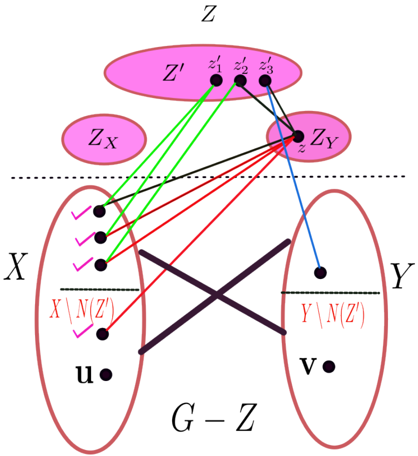


Reduction Rule 4

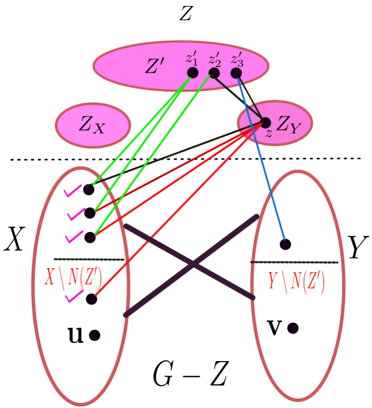
- Mark all vertices in $N(Z') \cap X$ and $N(Z') \cap Y$.



- For each vertex $z \in Z$, mark one of its non-neighbour (if it exists) each in $X \setminus N(Z')$ and $Y \setminus N(Z')$.



Reduction Rule 4



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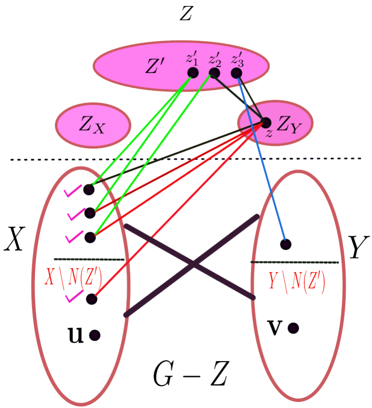

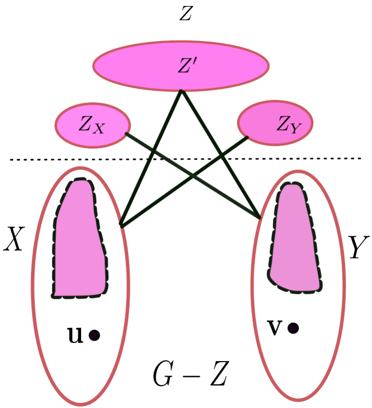

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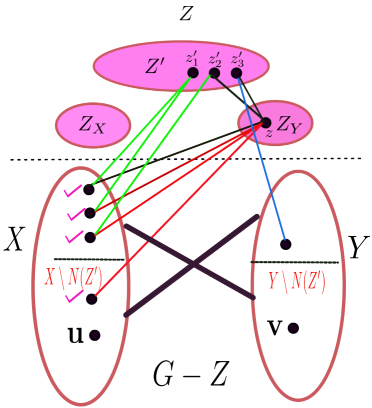
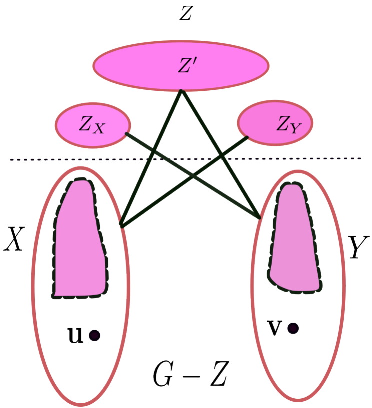
Reduction Rule 4





The marking procedure marks $\mathcal{O}(k^2)$ vertices.

Future Directions

1. Faster FPT Algorithm.
2. Linear vertex kernel for Bal Biclique Contraction.
3. Biclique Contraction: lower bound $\ell = n - k$ on the number of vertices in the resultant biclique is
   - W[1]-hard.
   - XP Algorithm?
4. Bal Biclique Contraction: when parameterized by ℓ
   - para-NP-hard ?

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NP-completeness of Biclique Contraction

- Red-Blue Dominating Set $\leq_P$ Biclique Contraction.
  - $(G, R, B, \kappa) \longmapsto (H, k)$ where H is bipartite, $k = |B| + \kappa$

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NP-completeness of Biclique Contraction

- Red-Blue Dominating Set $\leq_P$ Biclique Contraction.
  - $(G, R, B, \kappa) \longmapsto (H, k)$ where H is bipartite, $k = |B| + \kappa$
- Biclique Contraction is NP-complete even when restricted to bipartite graphs.
- Biclique Contraction does not admit any polynomial compression (or kernel)

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RED-BLUE DOMINATING SET

RED-BLUE DOMINATING SET

Given a bipartite graph G with bipartition $\langle R, B \rangle$ and an integer κ, the objective is to find a set $S \subseteq R$ of size at most κ that dominates B.

A set X is said to *dominate* a set Y if $Y \subseteq N(X)$.

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NP-completeness of BICLIQUE CONTRACTION

- Let (G, R, B, κ) be an instance of RED BLUE DOMINATING SET
- (H, k) be an instance of BICLIQUE CONTRACTION (H is bipartite)
- $k = |B| + \kappa$

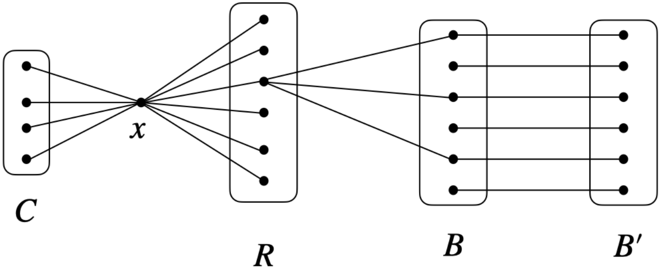

Figure 1: $|V(H)| = \mathcal{O}(|V(G)|)$, $C = \{v_1, \ldots, v_{\kappa+|B|+1}\}$

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NP-completeness of BALANCED BICLIQUE CONTRACTION

- HYPERGRAPH 2-COLORING $\leq_P$ BAL BICLIQUE CONTRACTION.

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NP-completeness of Balanced Biclique Contraction

- Hypergraph 2-Coloring $\leq_P$ Bal Biclique Contraction.
  - $(\mathcal{G}) \longmapsto (G, \kappa)$, κ is a function of the number of vertices and the number of hyperedges in $\mathcal{G}$

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NP-completeness of Balanced Biclique Contraction

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NP-completeness of BAL BICLIQUE CONTRACTION

HYPERGRAPH 2-COLORING problem

In the HYPERGRAPH 2-COLORING problem, the input is a hypergraph $\mathcal{G}$ and the objective is to determine if there is a 2-coloring $\phi : V(\mathcal{G}) \mapsto \{1, 2\}$ such that no hyperedge is monochromatic.

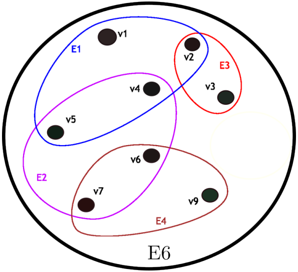

Figure 2: Hypergraph $\mathcal{G}$

NP-completeness of BAL BICLIQUE CONTRACTION

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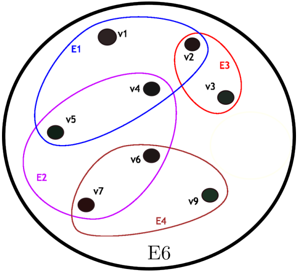

Figure 2: Hypergraph $\mathcal{G}$

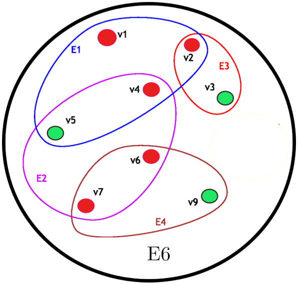

Figure 3: 2-coloring

NP-completeness of Bal Biclique Contraction

- Given an instance $(\mathcal{G})$ of Hypergraph 2-Coloring
- (H, k) be an instance of Bal Biclique Contraction.
- $k = 2M + N - 2$

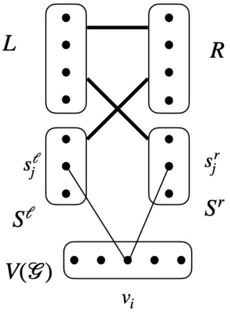

Figure 4: $|V(H)| = \mathcal{O}(|V(\mathcal{G})|)$.

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NP-completeness of BAL BICLIQUE CONTRACTION

(H, k) is a YES-instance of BAL BICLIQUE CONTRACTION
$\Leftrightarrow (G, \kappa = k + |Z|)$ is a YES-instance of BAL BICLIQUE CONTRACTION.

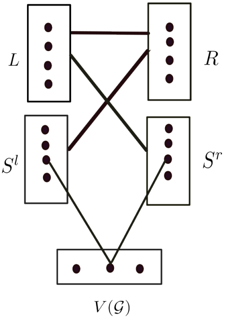

Figure 5: Intermediate graph H.

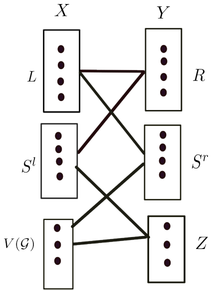

Figure 6: Bipartite graph G

References I

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Fixed-parameter algorithms for cluster vertex deletion.
Theory Comput. Syst., 47(1):196–217.