Revisiting Path Contraction and Cycle Contraction

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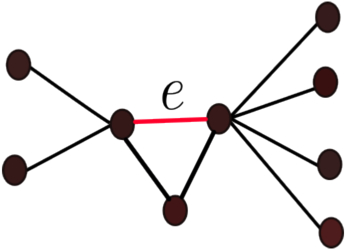
Edge Contractions



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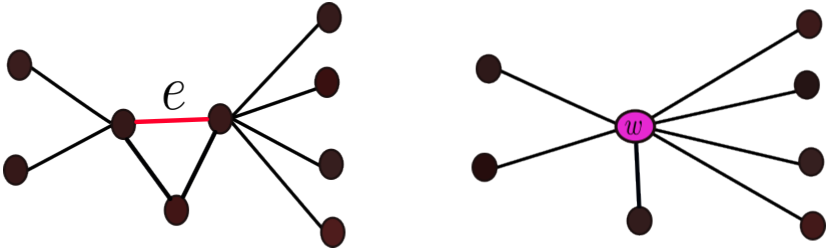
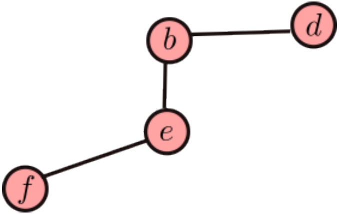
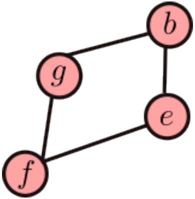


Path and Cycle

- P_ℓ denotes path on ℓ vertices



- C_ℓ denotes cycle on ℓ vertices



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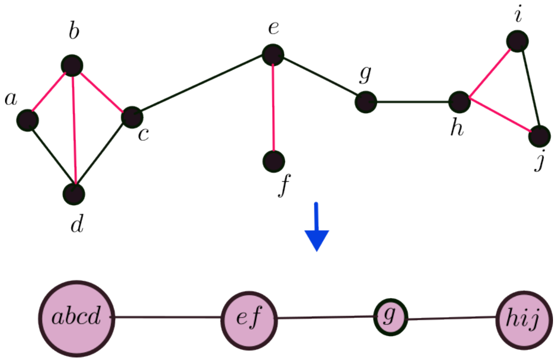
Path Contraction

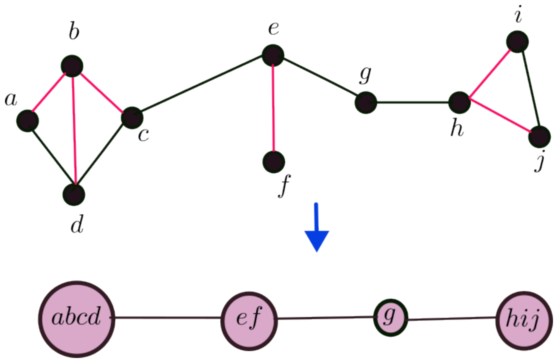
Path Contraction

Input: A connected undirected graph G on n vertices and an integer k.

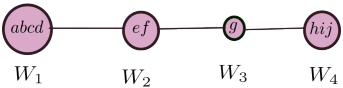
Question: Can one contract at most k edges in G to obtain a path?

Parameter: k.





P_4 witness structure is $(W_1, \ldots, W_4)$.



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Cycle Contraction

Cycle Contraction
Input: A connected undirected graph G on n vertices and an integer k.
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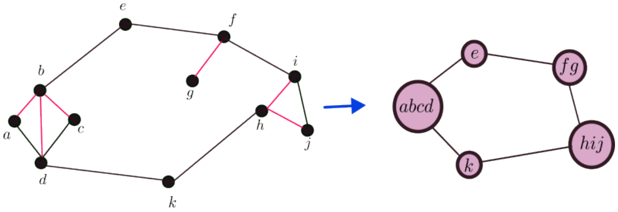
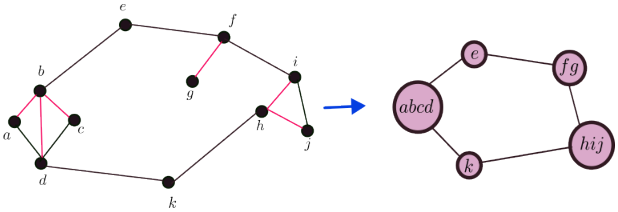
Parameter: k.

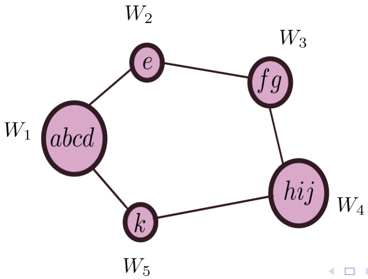


Cycle Contraction



C_5 witness structure $(W_1, \ldots, W_4, W_5)$

W_1 and W_5 are regarded as consecutive witness sets.



Path Contraction

- [Brouwer and Veldman, 1987]
  - NP-complete.
- [Heggernes et al., 2014a]
  - Exact algorithm running in $\mathcal{O}^*(2^n)$ time.
  - Kernel with at most $5k + 3$ vertices.
  - FPT algorithm running in $2^{k+\mathcal{O}(\sqrt{k}\log k)}$ time.
- [Li et al., 2017]
  - Improved kernel of at most $3k + 4$ vertices.
- [Agrawal et al., 2020]
  - Exact algorithm running in $\mathcal{O}^*(1.99987^n)$-time.

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Cycle Contraction

- [Brouwer and Veldman, 1987], [Hammack, 2002].
  - NP-complete.
- [Belmonte et al., 2014]
  - Kernel with at most $6k + 6$ vertices.
- [Sheng and Sun, 2019]
  - Improved kernel of at most $5k + 4$ vertices.

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Our Results

1. Path Contraction admits an algorithm running in $\mathcal{O}^*(2^k)$ time.

[1] [Heggernes et al., 2014b]There is an $\mathcal{O}(n^2 \cdot tw)$-time algorithm for Path Contraction on chordal graphs where tw is the treewidth of the input graph.

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Our Results

1. Path Contraction admits an algorithm running in $\mathcal{O}^*(2^k)$ time.
2. Cycle Contraction admits an algorithm running in $\mathcal{O}^*((2+\epsilon_\ell)^k)$ time where $0 < \epsilon_\ell \leq 0.5509$ and ϵ_ℓ decreases as ℓ increases.

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3. Longest cycle to which a graph can be contracted to can be solved in $\mathcal{O}^*(2.5191^n)$ time.

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4. Path Contraction on planar graphs admits a polynomial-time algorithm.

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3. Longest cycle to which a graph can be contracted to can be solved in $\mathcal{O}^*(2.5191^n)$ time.
4. PATH CONTRACTION on planar graphs admits a polynomial-time algorithm.
5. PATH CONTRACTION on chordal graphs does not admit an algorithm running in time $\mathcal{O}(n^{2-\epsilon} \cdot 2^{o(tw)})$ for any $\epsilon > 0$ under the Orthogonal Vectors Conjecture.[1]

[1] [Heggernes et al., 2014b]There is an $\mathcal{O}(n^2 \cdot tw)$-time algorithm for PATH CONTRACTION on chordal graphs where tw is the treewidth of the input graph.

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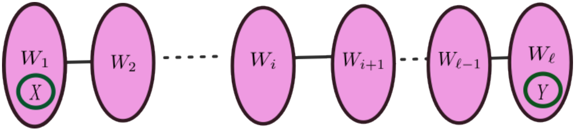
PATH CONTRACTION WITH CONSTRAINED ENDS
Input: A connected graph G, two disjoint subsets $X, Y \subseteq V(G)$ and an integer k.
Question: Can one contract at most k edges in G to obtain a path with witness structure $(W_1, \dots, W_\ell)$ such that $X \subseteq W_1$ and $Y \subseteq W_\ell$?
Parameter: k.



Path Contraction

Theorem

Path Contraction With Constrained Ends *admits an algorithm running in time* $\mathcal{O}^*(2^{k-|X|-|Y|})$.

Observe that Path Contraction With Constrained Ends when $X = \emptyset$ and $Y = \emptyset$ is Path Contraction.

Corollary

Path Contraction *admits an algorithm running in time* $\mathcal{O}^*(2^k)$.

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Path Contraction With Constrained Ends

Contracting to path of length at least 6. i.e. $k \leq n - 6$.

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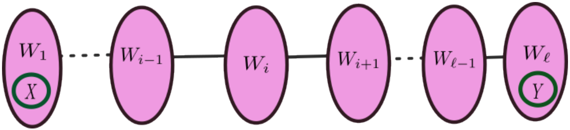

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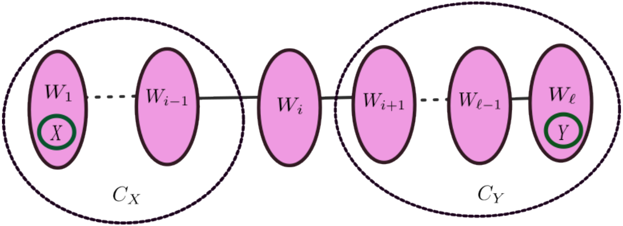
Path Contraction With Constrained Ends



Each non-terminal witness set W_i is a connected set with $W_i \subseteq V(G) \setminus (X \cup Y)$

1. $G - W_i$ has exactly two connected components C_X, C_Y containing X and Y, respectively.
2. $|(N(W_i) \setminus (X \cup Y)| + |W_i| \leq k + 5 - |X| - |Y|$

For any $2 \leq i \leq \ell - 1$
$|W_1| - 1 + |W_{i-1}| - 1 + |W_i| - 1 + |W_{i+1}| - 1 + |W_\ell| - 1 \leq k$

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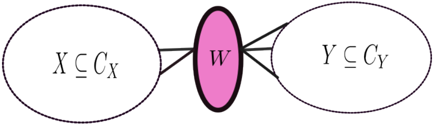
Potential k-witness Set

A connected set $W \subseteq V(G) \setminus (X \cup Y)$:

1. $G - W$ has exactly two connected components C_X, C_Y containing X and Y, respectively.
2. $|(N(W) \setminus (X \cup Y)| + |W| \leq k + 5 - |X| - |Y|$



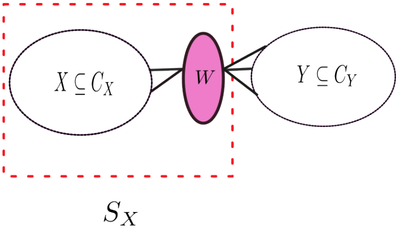
Enumerating Potential k-witness sets

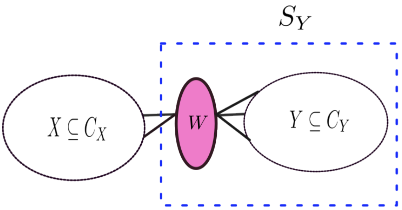
The number of potential k-witness sets in G is $\mathcal{O}^*(2^{k-|X|-|Y|})$ and these sets can be enumerated in $\mathcal{O}^*(2^{k-|X|-|Y|})$ time.

Path Contraction With Constrained Ends

Potential k-prefix set

For a potential k-witness set W, the sets $S_X := C_X \cup W$ and $S_Y := C_Y \cup W$ are called the *potential k-prefix sets* associated with W.





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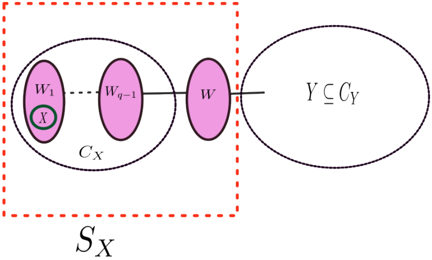

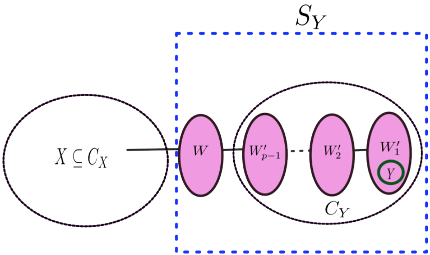

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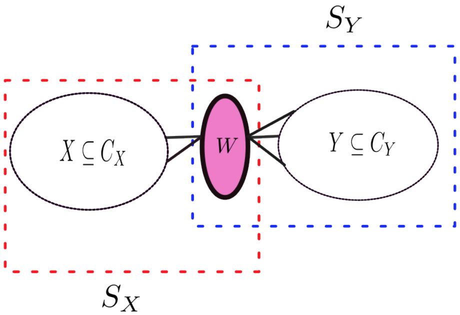
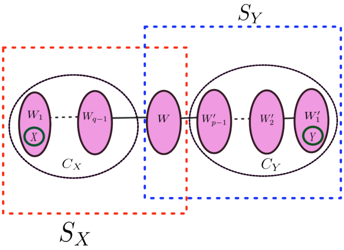


Using Dynamic Programming



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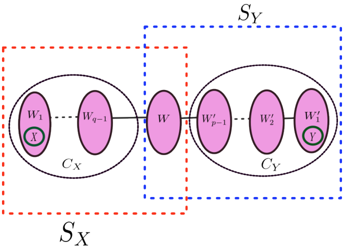

$\Gamma[W, S_X, k'] = \texttt{True}$ if and only if $G[S_X]$ is k'-contractible to a path
with witness structure $(W_1, \ldots, W_q = W)$
for some $q \geq 2$ such that $X \subseteq W_1$

$\Gamma[W, S_Y, k''] = \texttt{True}$ if and only if $G[S_Y]$ is k''-contractible to a path
with witness structure $(W'_1, \ldots, W'_p = W)$
for some $p \geq 2$ such that $Y \subseteq W'_1$

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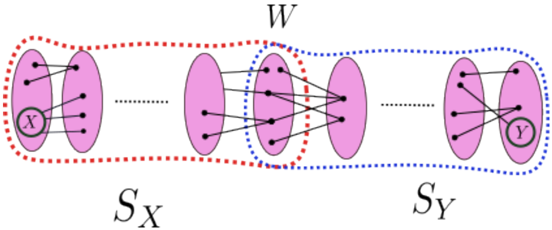
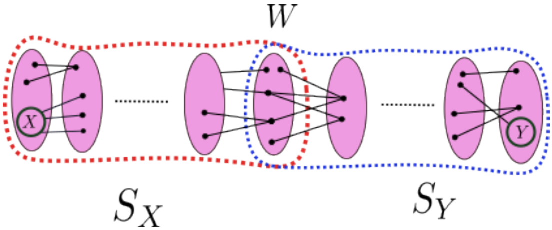
Using Dynamic Programming



Using Dynamic Programming



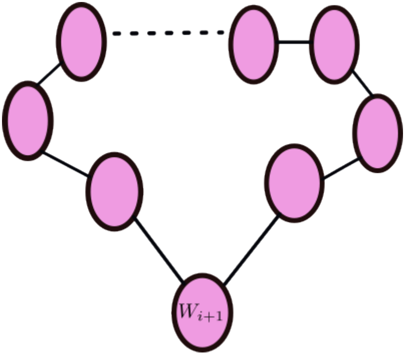
(G, X, Y, k) is an YES instance if and only if $\exists W, k_1, k_2$ such that

- $\Gamma[W, S_X, k_1] = \texttt{True}$
- $\Gamma[W, S_Y, k_2] = \texttt{True}$
- $k_1 + k_2 - (|W| - 1) \leq k$

Theorem

PATH CONTRACTION WITH CONSTRAINED ENDS *admits an algorithm running in time* $\mathcal{O}^*(2^{k-|X|-|Y|})$.

Cycle Contraction



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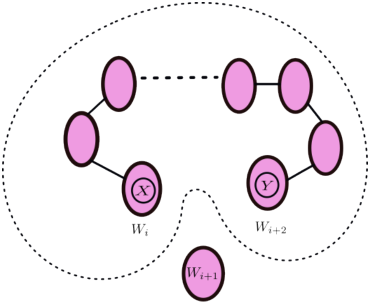

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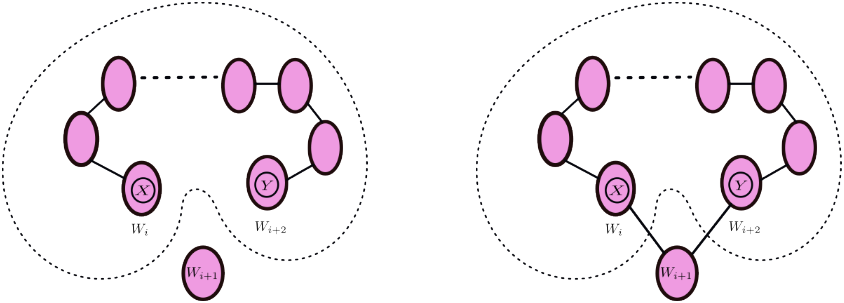

Once W_{i+1}, X and Y are known - solve a Path Contraction With Constrained Ends instance.

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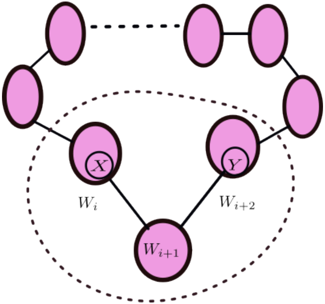
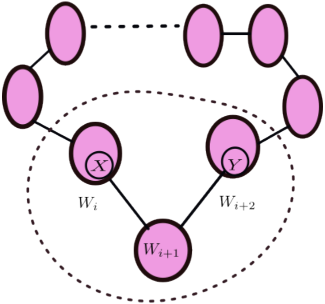
Observation

If G is k-contractible to C_ℓ where $\ell \geq 4$ with witness structure $(W_1, W_2, \ldots, W_\ell)$, then there is a set of three consecutive witness sets W_i, W_{i+1}, W_{i+2} such that $|W_i| + |W_{i+1}| + |W_{i+2}| \leq 3 + \frac{3k}{\ell}$.



The number of choices of (X, W_{i+1}, Y) is $\mathcal{O}^*(3^{\frac{3k}{\ell}})$ where $N(W_{i+1}) = X \cup Y$ and can be enumerated in same time.

Cycle Contraction



Theorem

Cycle Contraction runs in $\mathcal{O}^*((2+\epsilon_\ell)^k)$ time where $0 < \epsilon_\ell \leq 0.5509$ and ϵ_ℓ decreases as ℓ increases.

Future Directions

Path Contraction and Cycle Contraction

1. $\mathcal{O}^*((2-\epsilon)^k)$-time algorithm for Path Contraction where $\epsilon > 0$
2. $\mathcal{O}^*(2^k)$-time algorithm for Cycle Contraction
3. Parameterization by treewidth
4. Faster algorithms in special graph classes

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