A Finer View of the Parameterized Landscape of Labeled Graph Contractions

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Labeled Contractions

Definition (Labeled contraction)

Let G be a graph and let $uv \in E(G)$. The *labeled contraction* (u, v) produces a graph $G/(u, v)$ by:

- adding edges between u and every vertex in $N_G(v) \setminus N_G[u]$;
- deleting vertex v and all its incident edges.

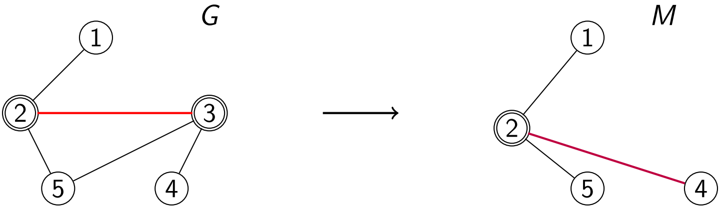

Figure 1: Contraction along edge $(2, 3)$

Problem Definition

Definition (Labeled Contractibility)

Input: Two graphs G and H with $V(H) \subseteq V(G)$.
Question: Does there exist a sequence of contractions from graph G to H?

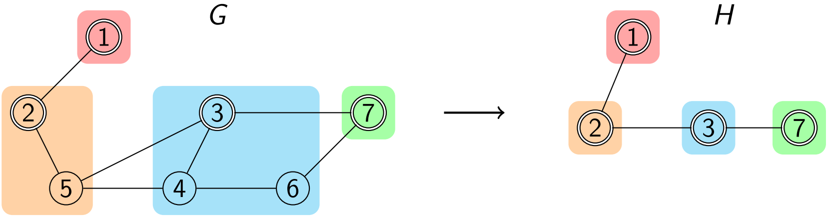

Figure 2: Shaded regions in G form the witness sets, each containing exactly one representative vertex (circled). Contracting each witness set yields H.

Background

Graph modification problems have played a central role in the development of parameterized complexity; see the survey by Crespelle et al. [3].

Graph Modification Problems. Transform a graph G into a graph belonging to a class $\mathcal{F}$ using a bounded number k of allowed operations: vertex/edge deletion, edge addition, or edge contraction.

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Graph Modification Problems. Transform a graph G into a graph belonging to a class $\mathcal{F}$ using a bounded number k of allowed operations: vertex/edge deletion, edge addition, or edge contraction.

- Such problems are **NP-hard** for many natural graph classes $\mathcal{F}$.
- For singleton classes $\mathcal{F} = \{H\}$:
  - Vertex/edge deletion and edge addition are polynomial-time solvable.
  - **Edge contraction is NP-hard** [1, 2].

Parameterized Landscape

Lafond and Marchand initiated the systematic study of the parameterized complexity of **edge contraction problems** on *uniquely labeled graphs*.

"The Parameterized Landscape of Labeled Graph Contractions"

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- In particular, showed that even for labeled graphs, **edge contraction is W[1]-hard when parameterized by** k.

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- This stands in sharp contrast to vertex/edge deletion or edge addition, which are FPT whenever $\mathcal{F}$ admits a finite forbidden structure.

Our work: *A Finer View of the Parameterized Landscape of Labeled Graph Contractions* addresses and refines questions raised in this framework.

Existing Result and Our Contribution

Theorem (In parameter of tree-width, WADS'25 [4])

LABELED CONTRACTIBILITY *is fixed parameter tractable when parametererized by* $\mathsf{tw} = \max(\mathsf{tw}(G), \mathsf{tw}(H))$.

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- *admits an algorithm running in time* $2^{\mathcal{O}(\mathrm{tw}^2)} \cdot |V(G)|^{\mathcal{O}(1)}$.
- *does not admit an algorithm running in time* $2^{o(\mathrm{tw}^2)} \cdot |V(G)|^{\mathcal{O}(1)}$, *unless the ETH fails.*

A Rare Complexity dependence on Treewidth

- The majority of FPT algorithms on treewidth usually admit $(2^{\mathcal{O}(\mathrm{tw})})$, $(2^{\mathcal{O}(\mathrm{tw}\log \mathrm{tw})})$ or $(2^{2^{\mathrm{tw}}})$ algorithms. But, our result establishes a quadratic dependence on tw, $(2^{\mathcal{O}(\mathrm{tw}^2)})$, which represents a much rarer complexity barrier in FPT.
- The only other problems to our knowledge exhibiting *super-exponential* dependence on treewidth are [7, 8, 9].
- Moreover, [10, 11, 12] exhibit *single-exponential but polynomial* dependence on pathwidth. Related phenomena for parameterization by vertex cover are reported in [6, 13, 14, 15].

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Labeled Contractibility *is NP-hard even when the degeneracy $\delta(G)$ is bounded.*

Theorem (Strengthened Result)

Labeled Contractibility *is NP-hard even when the maximum degree $\Delta(G)$ is bounded.*

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Theorem (Strengthened Result)

LABELED CONTRACTIBILITY *is NP-hard even when the maximum degree* $\Delta(G)$ *is bounded.*

These results suggest that tractability is unlikely when parameterizing solely by the solution size k or the degeneracy $\delta(G)$. Nevertheless, Lafond and Marchand showed the problem becomes tractable for the combined parameter $(k + \delta(G))$

Existing Result and Our Contribution

Theorem (In parameter $k + \delta(G)$, WADS'25 [4])

LABELED CONTRACTIBILITY *is fixed parameter tractable when parametererized by $(k + \delta(G))$ with running time $(\delta(G) + 2k)^k n^{O(1)}$.*

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LABELED CONTRACTIBILITY *is fixed parameter tractable when parametererized by $(k + \delta(G))$ with running time $(\delta(G) + 2k)^k n^{O(1)}$.*

- We replace the bound $\delta(H) \leq \delta(G) + k$ with the tighter

$$\delta(H) \leq \delta(G) \cdot \frac{|V(G)|}{|V(G)| - k}.$$

- If $|V(G)| \geq (1 + \epsilon)k$, then $\delta(H) \leq \delta(G) \cdot c_\epsilon$ for a constant c_ϵ, yielding a faster algorithm.

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Theorem (Improved bound)

LABELED CONTRACTIBILITY *admits an algorithm running in time $(\delta(H) + 1)^k \cdot |V(G)|^{O(1)}$.*

Existing Result and Our Contribution

Theorem (Improved bound)

LABELED CONTRACTIBILITY *admits an algorithm running in time* $2^{\mathcal{O}(k)} \cdot |V(G)|^{O(1)}$*, when degeneracy of* G *is a constant.*

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Open Question (WADS 2025 [4]). The existence of a sub-exponential time algorithm for LABELED CONTRACTIBILITY parameterized by k when $\delta(G)$ was left open.

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Open Question (WADS 2025 [4]). The existence of a sub-exponential time algorithm for Labeled Contractibility parameterized by k when $\delta(G)$ was left open.

Theorem (Our Contribution)

Assuming ETH, no algorithm can solve Labeled Contractibility *in time* $2^{o(|V(G)|+|V(H)|)}$ *(and hence in time* $2^{o(k)} \cdot |V(G)|^{\mathcal{O}(1)}$*), when* $\delta(G)$ *is constant.*

Existing Result and Our Contribution

- Provided an algorithm running in time in time $|V(H)|^{\mathcal{O}(|V(G)|)}$ that iterates over all the possible functions from G to H and showed the optimality of the same.

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Theorem (Brute-force)

LABELED CONTRACTIBILITY *cannot be solved in time* $|V(H)|^{o(|V(G)|)}$ *under the Exponential Time Hypothesis (ETH).*

Conditional Lower Bound

Sub-Cubic Partitioned Vertex Cover (Sub-Cubic PVC)

Input: Sub-cubic graph G partitioned into $\mathcal{P} = \{C_1, \ldots, C_t\}$, and budget k_i for each C_i.

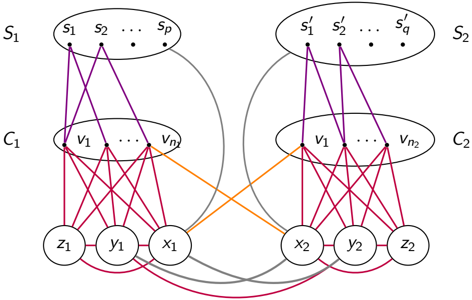
Question: Find a Vertex Cover X such that $|X \cap C_i| \leq k_i$ for all i.

- **Hardness Basis [6]:** SUB-CUBIC PVC does not admit an algorithm running in time $2^{o(n)}$ unless ETH fails.
- **Input Structure Used:** The instance satisfies $t = \mathcal{O}(\sqrt{n})$ and $|C_i| = \mathcal{O}(\sqrt{n})$, which is key to bounding the treewidth.

We reduce an instance $(G, \{C_i\}, \{k_i\})$ of SUB-CUBIC PVC to an instance (G', H') of LABELED CONTRACTIBILITY.

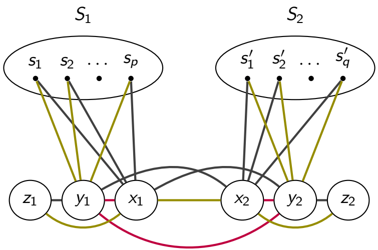
Construction of G'

- Contains all vertices of G, along with auxiliary vertices.
- For each clause C_i, we introduce auxiliary vertices $\{x_i, y_i, z_i\}$ and a set S_i.
- Each vertex in S_i represents a $(k_i + 1)$-subset of C_i.

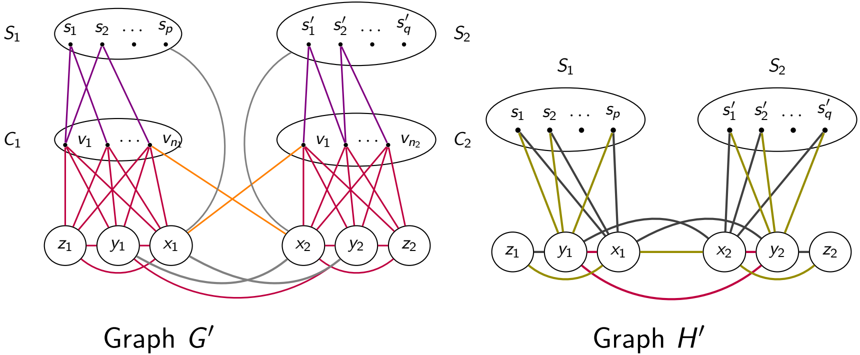


Construction of H'

- Vertex set defined as $V(H') = V(G') \setminus V(G)$.
- For each i, every vertex $s \in S_i$ is adjacent to y_i.
- All vertices x_i form a clique in H'.
- These edges are *present in H' but absent in G'*.



Correctness of the Reduction



(⇒) **Reduction Rule**

Given a PVC solution $X \subseteq V(G)$,
contract (a, x_i) for all $a \in X \cap C_i$, and
contract (b, y_i) for all $b \in C_i \setminus X$.

Correctness of the Reduction

($\Rightarrow$) Reduction Rule

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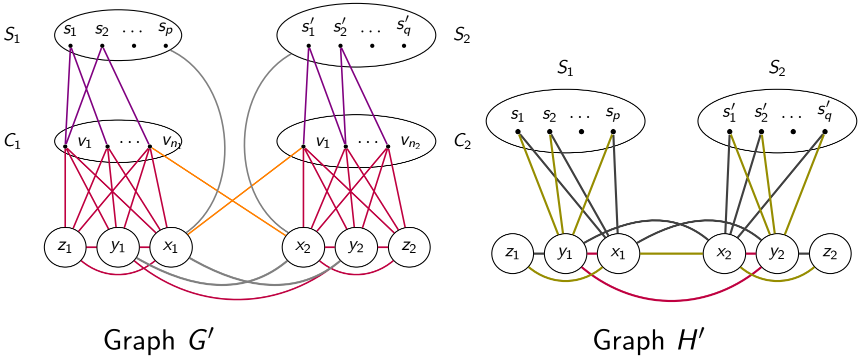
contract (a, x_i) for all $a \in X \cap C_i$, and
contract (b, y_i) for all $b \in C_i \setminus X$.

- For every $i \neq j \in [t]$, there exists one edge in G between C_i and C_j. For $(u, v) \in E(G)$, either u or v is in vertex cover X. This guarantees that the vertices $\{x_1, \ldots, x_t\}$ form a clique.
- For each $i \in [t]$, the budget constraint $|X \cap C_i| \leq k_i$ implies that no subset of C_i of size $k_i + 1$ is entirely contracted to x_i.
- Hence, at least one vertex in this subset must be contracted to y_i, ensuring the presence of the edge (s, y_i).

Correctness of the Reduction



($\Leftarrow$) Construction of the PVC Solution

From a contraction sequence transforming G' into H', define $X \subseteq V(G)$ by including, for each $i \in [t]$, all vertices in C_i that are contracted to x_i.

Correctness of the Reduction

($\Leftarrow$) Construction of the PVC Solution

From a contraction sequence transforming G' into H', define $X \subseteq V(G)$ by including, for each $i \in [t]$, all vertices in C_i that are contracted to x_i.

- Since $V(H') = V(G') \setminus V(G)$, every contraction involves a vertex from some block C_i. Further by construction, vertices in C_i can only be contracted to x_i or y_i.
- Each vertex $s \in S_i$ represents a subset $C'_i \subseteq C_i$ of size $k_i + 1$ and is adjacent to y_i in H'. To preserve the edge (s, y_i), at least one vertex of C'_i must be contracted to y_i.

Conclusion: $|X \cap C_i| \leq k_i$ for all $i \in [t]$.

Correctness and Hardness

- Consider an edge $(u, v) \in E(G)$ with $u \in C_i$ and $v \in C_j$, $i \neq j$.
- Since x_i and x_j are adjacent in H', and no edge (x_i, x_j) exists in G', at least one of u or v must be contracted to x_i or x_j, respectively. Thus, either $u \in X$ or $v \in X$, and every edge of G is covered.

Conclusion: X is a valid vertex cover of G.

- The reduction is polynomial-time and parameter-preserving.
- Since Sub-Cubic PVC is ETH-hard, the hardness carries over to Labeled Contractibility.

Maximum Common Labeled Contraction

Definition (MCLC)

- **Input:** Two vertex-labeled graphs G and H, and an integer k.
- **Question:** Do there exist contraction sequences S_1 for G and S_2 for H such that

$$G/S_1 = H/S_2 \quad \text{and} \quad |S_1| + |S_2| \leq k\,?$$

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Key Observations

- The problem is equivalent to finding a *common labeled contraction* of G and H of size at least $|V(H)|$.
- Consequently, complexity results for LABELED CONTRACTIBILITY directly transfer to **Maximum Common Labeled Contraction**.

Open Problems

- Extend DP framework to MAXIMUM COMMON LABELED CONTRACTION.
- Explore FPT algorithms for feedback vertex set or vertex cover number.
- Study practical implementations or heuristics for large/dense graphs.

THANK YOU!

Any Questions?

References I

T. Watanabe, T. Ae, A. Nakamura.
On the NP-hardness of edge-deletion and -contraction problems.
Discrete Applied Mathematics, 6(1):63–78, 1983.

T. Asano, T. Hirata.
Edge-Contraction Problems.
Journal of Computer and System Sciences, 26(2):197–208, 1983.

C. Crespelle, P. G. Drange, F. V. Fomin, P. A. Golovach.
A survey of parameterized algorithms and the complexity of edge modification.
arXiv:2001.06867, 2020.

M. Lafond, B. Marchand.
The Parameterized Landscape of Labeled Graph Contractions.
In *WADS 2025*, LIPIcs, 2025.

References II

B. Marchand, N. Tahiri, O. Tremblay Savard, M. Lafond.
Finding Maximum Common Contractions Between Phylogenetic Networks.
In *WABI 2024*, LIPIcs, 2024.

A. Agrawal, D. Lokshtanov, S. Saurabh, M. Zehavi.
Split Contraction: The Untold Story.
ACM TOCT, 11(3), 2019.

D. Chakraborty, F. Foucaud, D. Majumdar, P. Tale.
Tight (Double) Exponential Bounds for Identification Problems.
In *ISAAC 2024*, LIPIcs.

F. Foucaud et al.
Problems in NP Can Admit Double-Exponential Lower Bounds.
In *ICALP 2024*, LIPIcs.

References III

P. Tale.
Double Exponential Lower Bound for Telephone Broadcast.
arXiv:2403.03501, 2024.

I. Sau, U. dos Santos Souza.
Hitting Forbidden Induced Subgraphs on Bounded Treewidth Graphs.
Information and Computation, 281, 2021.

M. Pilipczuk.
Problems Parameterized by Treewidth Tractable in Single Exponential Time.
In *MFCS 2011*.

M. Cygan, D. Marx, M. Pilipczuk, M. Pilipczuk.
Hitting Forbidden Subgraphs in Bounded Treewidth Graphs.
Information and Computation, 256, 2017.

References IV

B. Bergougnoux, T. Korhonen, J. Nederlof.
Tight Lower Bounds for Rank-width.
arXiv:2210.02117, 2022.

D. Chakraborty et al.
Parameterized Complexity of Isometric Path Partition.
arXiv:2508.05448, 2025.

F. Foucaud et al.
Metric Dimension and Geodetic Set Parameterized by Vertex Cover.
In *STACS 2025*, LIPIcs.