Algorithms and Hardness for Geodetic Set on Tree-like Digraphs

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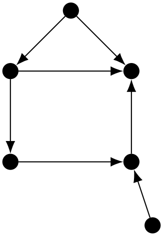
The Geodetic Set Problem



Definition: Geodetic set

A set S of vertices is a **geodetic set** in digraph $D = (V, A)$ if all vertices of $V \setminus S$ are lying on some shortest directed path whose endpoints are vertices of S.

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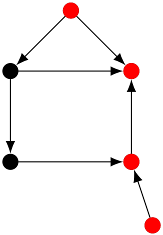

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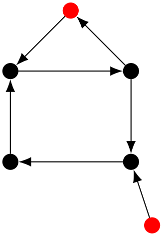

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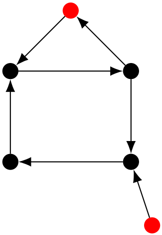

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GEODETIC SET is the problem of finding a geodetic set of minimum size in a given graph.

Known results

Undirected Extensively studied:

- NP-hard on **interval graphs** or **planar graphs of bounded degree** (Chakraborty et al., 2020).
- Polynomial time algorithms on **split graphs** (Douthat and Kong, 1995), **outerplanar graphs** (Mauro Mezzini, 2018).
- Several parameterized (in)tractability results.

Directed Few results known:

- NP-hard on **DAGs** whose underlying graph is **bipartite, co-bipartite or split**.
- Polynomial time algorithm on **oriented cacti** (Araújo and Arraes 2020).

GEODETIC SET, forests and motivations

- Extremal vertices are mandatory in any geodetic set.
  - In undirected graphs: **leaves**
  - In directed graphs: **sources and sinks**
- For forests, it is **optimal** (any vertex is either a leaf or on a path between two leaves)

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Goal : study tractability of Geodetic Set on instances "**close to**" **unoriented forests**. We consider:

- Directed trees (with digons)
- DAGs of bounded feedback vertex number
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Quantify distance to a forest: fvn and fen

Feedback vertex number: minimum number of vertices to remove so that the underlying graph contains no cycle

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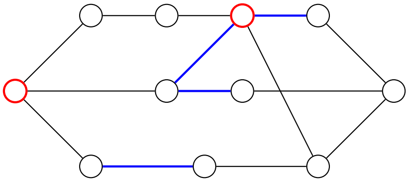

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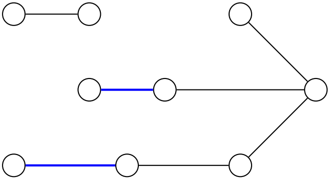

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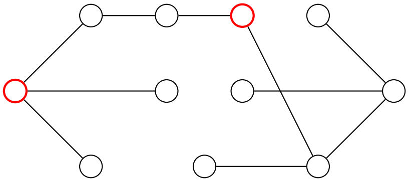

Parameterized complexity

Definition: Parameterized tractability

For a problem Π parameterized by k, we say that Π is **Fixed Parameter Tractable** (FPT) with respect to k if there exists an algorithm solving this problem on any instance (x, k) in time $f(k)\mathrm{poly}(|x|)$.

Here, we will consider fvn and fen as parameters for the GEODETIC SET.

Parameterized complexity of the undirected setting

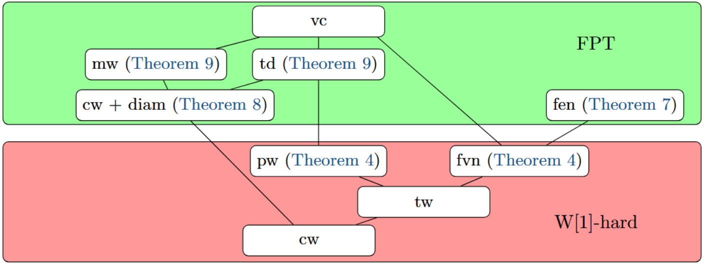

Figure 2: An overview of our results for GEODETIC SET, containing the parameters vertex cover number (vc), modular-width (mw), tree-depth (td), clique-width (cw), diameter (diam), feedback edge number (fen), path-width (pw), feedback vertex number (fvn) and tree-width (tw). An edge between two parameters indicates that the one below is smaller than some function of the other.

Kellerhals and Koana (2022)

Positive results

Theorem: Directed Trees (with digons)

GEODETIC SET on digraphs (possibly with digons) whose underlying graph is a tree is solvable in polynomial time

Theorem: DAGs of bounded fen

GEODETIC SET on DAGs without digons, whose underlying undirected graph has feedback edge number fen, admits an algorithm with running time $2^{\mathcal{O}(\text{fen})} \cdot n^{\mathcal{O}(1)}$, where n is the number of vertices in digraph.

Hardness result

Theorem: DAGs of bounded fvn

GEODETIC SET is NP-hard, even when restricted to DAGs whose underlying graph has feedback vertex number equal to 12.

Directed trees

Directed Trees and Extremal Sets

Optimal geodetic sets are:

- In trees $\rightarrow$ leaves

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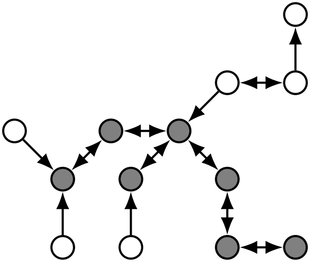
Definition: Extremal set

A strongly connected component of S is called an **extremal set** if either $N^{-}(S) \setminus S = \emptyset$ or $N^{+}(S) \setminus S = \emptyset$



Extremal sets property

Proposition

Let $\mathcal{E}$ be an extremal set of some digraph D and S a geodetic set of D Then, $S \cap V(\mathcal{E}) \neq \emptyset$.

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Let $\mathcal{E}$ be an extremal set of some digraph D containing no leaf, and S a geodetic set of D, such that there exist a vertex $v \in E \cup S$. Then for any vertex $v' \in \mathcal{E} - v$, $S - v + v'$ is a geodetic set.

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$\rightarrow$ Any non-leaf vertex of an extremal set is equivalent in a solution, and any solution hit each extremal set!

Contract extremal sets

Contract all extremal sets:

- If one contains a leaf, contract and keep a leaf
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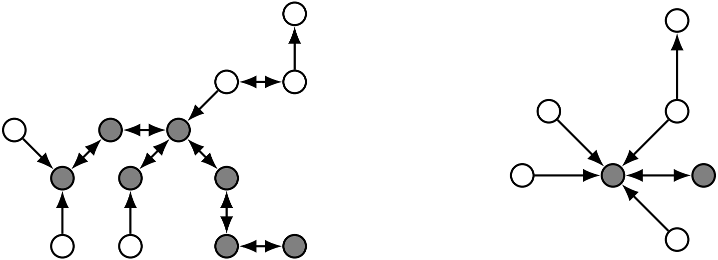
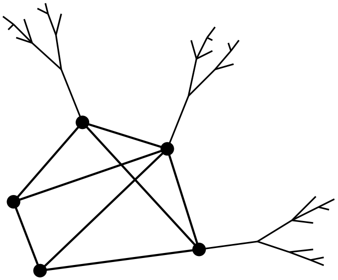


FPT and fen

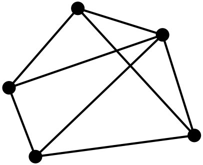
Core graph and associated properties

- "Hanging trees" are not dificult to handle



Core graph and associated properties

- "Hanging trees" are not dificult to handle
- What remains after trimming is a **core graph** with:
  - $\mathcal{O}(\text{fen})$ vertices of degree ≥ 3 (core vertices)
  - $\mathcal{O}(\text{fen})$ paths linking those vertices (core paths)



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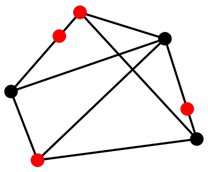
What to do with vertices on the paths?



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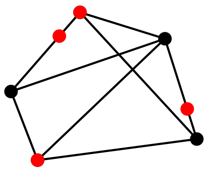

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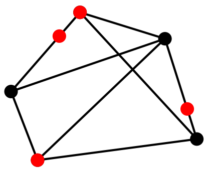


Handling the core paths

Proposition

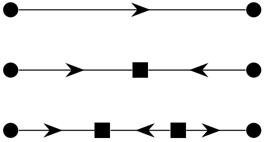
An optimal geodetic set contains at most 1 inner non-extremal vertex on each core path.

Case analysis on the number of extremal vertices inside the path

No extremal vertex

One extremal vertex

Two extremal vertices



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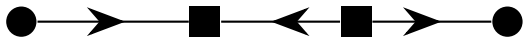
Two extremal vertices 

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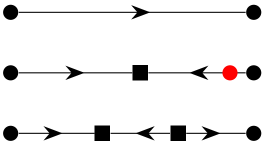

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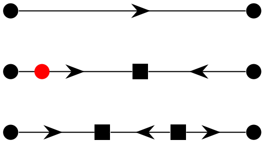

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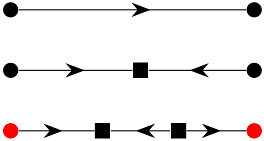


Hardness on digraphs of bounded fvn

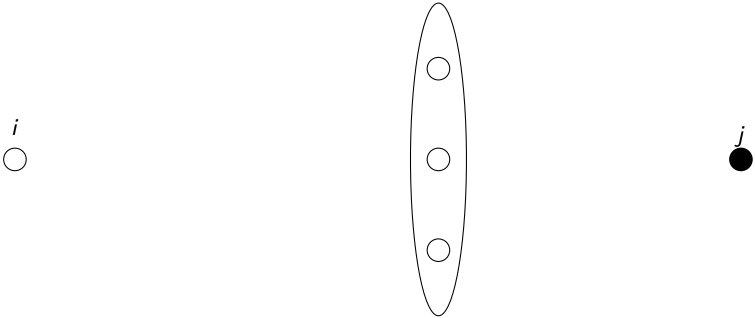
3D Matching

- A set X of $3n$ elements partitionned in 3 sets X^{α} X^{β} and X^{γ}, each of them of size n.
- A set of edges $E \subset X^{\alpha} \times X^{\beta} \times X^{\gamma}$.
- Goal : decide if there exists a set S of n edges such that any element of X is covered by one edge of S.

Theorem: Karp, 1972

3D Matching is NP-complete.

Adjacency gadget



Gadget encoding adjacency : all vertices are covered if and only if $i = j$.

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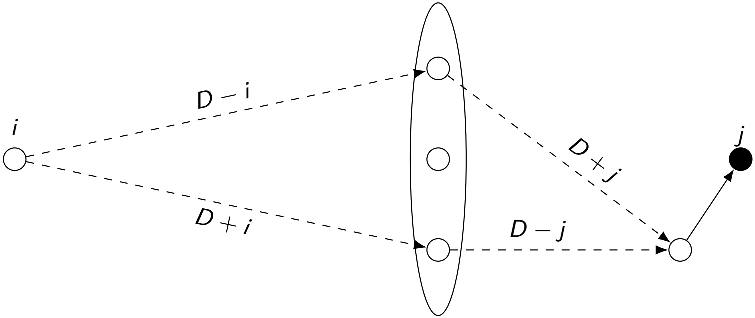

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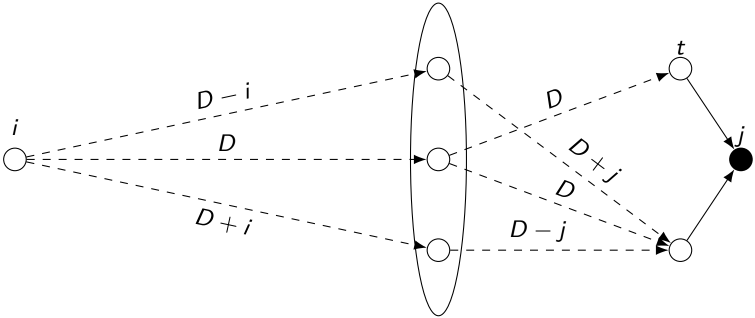

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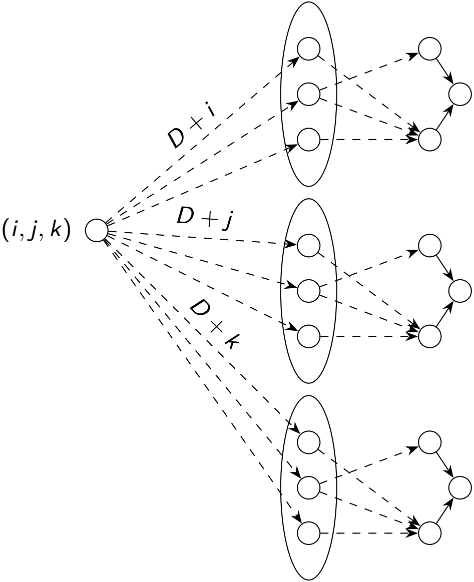
Encoding edges for 3D Matching



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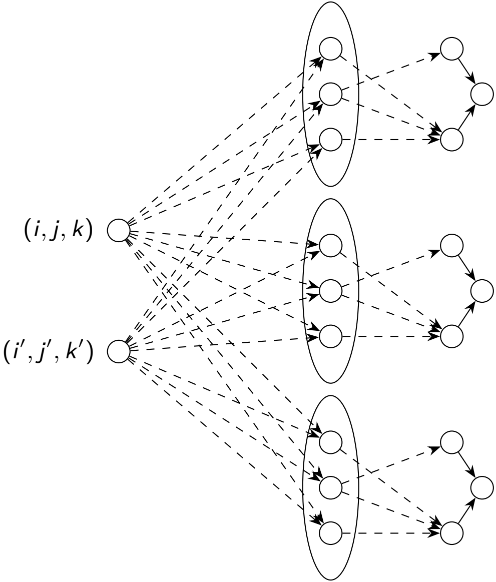

Further Work

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- Parameterized algorithms / intractability for other parameters.

Thank you!