Parameterized Complexity of Isometric Path Partition: Treewidth and Diameter

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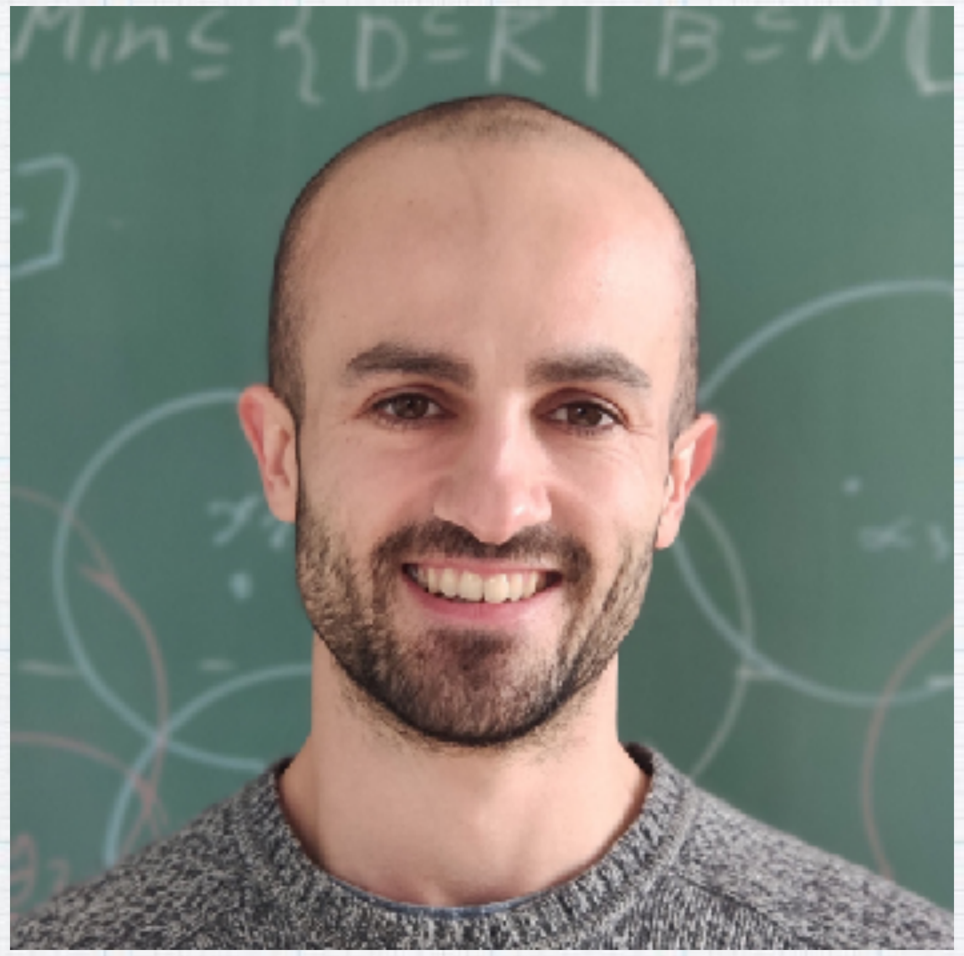

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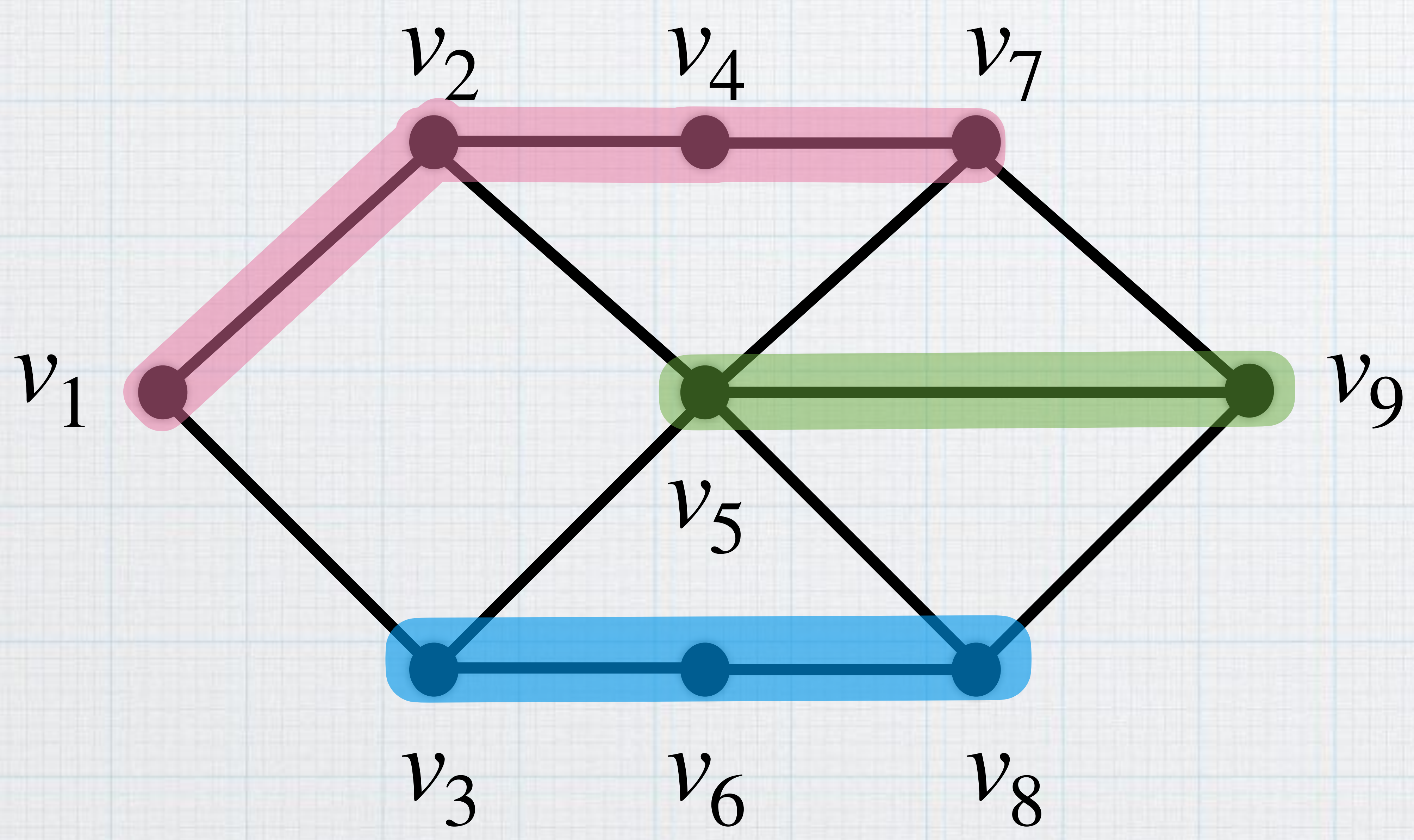
Isometric Path Partition

Input: Graph G, and integer k.

Output: Is there a partition of the vertex set of G into k sets, each of them forming an isometric (i.e. shortest) path in G?



- One of the metric-based optimization graph problem: where the solution is defined using some metric-based property of the graph (often the shortest distance).

- Metric-based optimization graph problem paramaterized by tree-width.

Q: Can we apply Courcelle's theorem?

A: No, because there is no MSO formula to encode: Given a subset X and two specified vertices s, t, does X form a shortest path between s and t? [1]

- Other metric-based problems like Metric Dimension, Geodetic Set, Strong Metric Dimension, etc are known to be para-NP-hard when parameterized by treewidth.

[1] Denis Kuperberg. Public communication on the Theoretical Computer Science Stack Exchange.
https://cstheory.stackexchange.com/a/52509, 2023.

Theorem 1. Isometric Path Partition admits an algorithm running in time $n^{\mathcal{O}(\mathrm{tw})}$.

- Improves results by Dumas et al. [SIDMA’24], Fernau et al [TCS’25].
- Dumas et al. [SIDMA’24] proved that every Yes-instance of Isometric Path Partition has treewidth bounded by an exponential function of the solution size.
- By Courcelle’s Theorem, they obtained an XP algorithm parameterized by the solution size.
- Baste et al. [STACS’25] improved the upper bound on treewidth by solution size by Dumas et al. [SIDMA’24] from exponential to polynomial.

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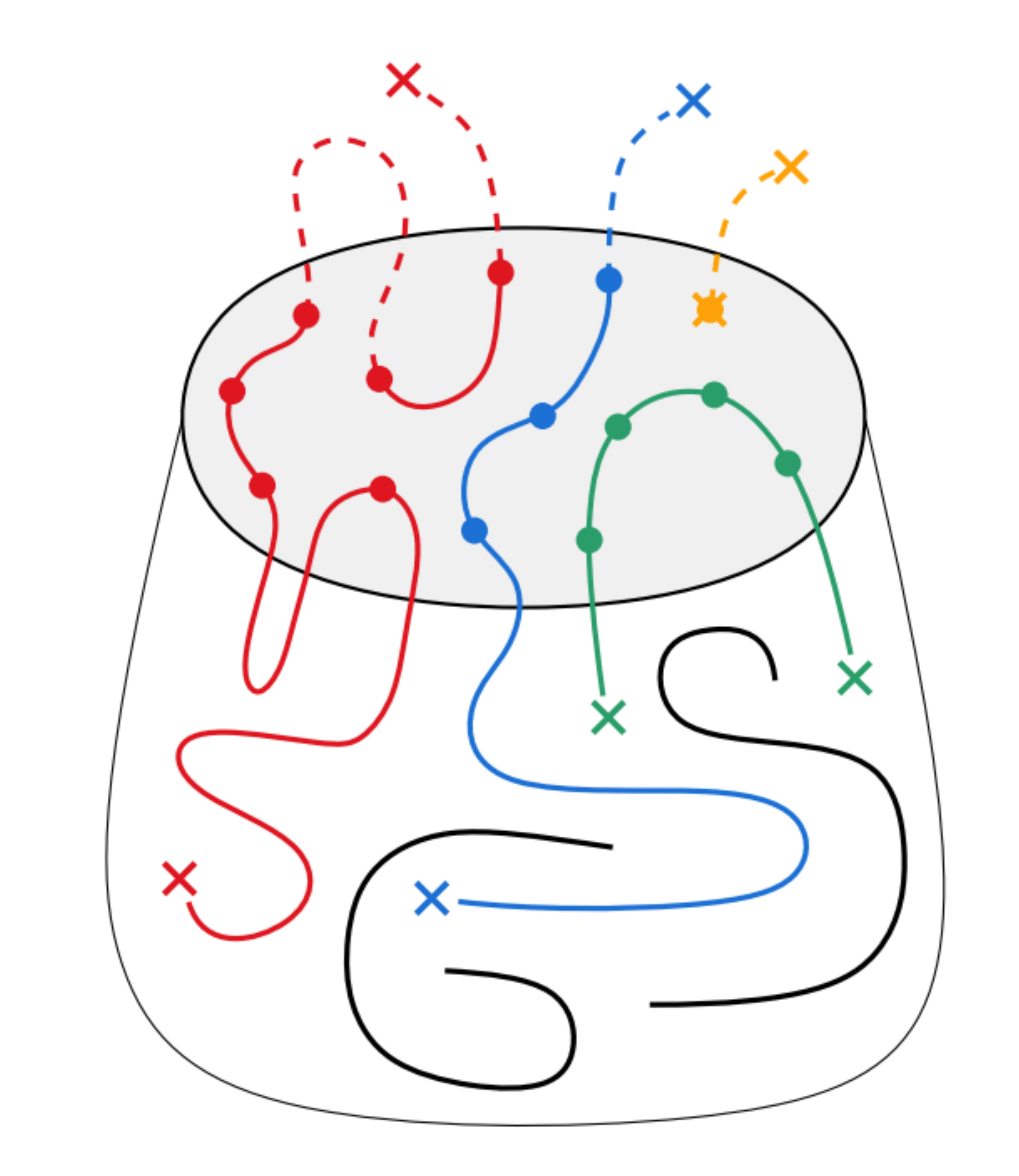

- At any given bag, at most tw many solution paths crosses it (because we are looking for partition).
- For each bag, the algorithm stores:
  - Possible intersections of the isometric paths with the bag ($\text{tw}^{\mathcal{O}(\text{tw})}$)
  - How these path pieces are connected outside bag. ($\text{tw}^{\mathcal{O}(\text{tw})}$)
  - The locations of the path endpoints. ($n^{\mathcal{O}(\text{tw})}$)

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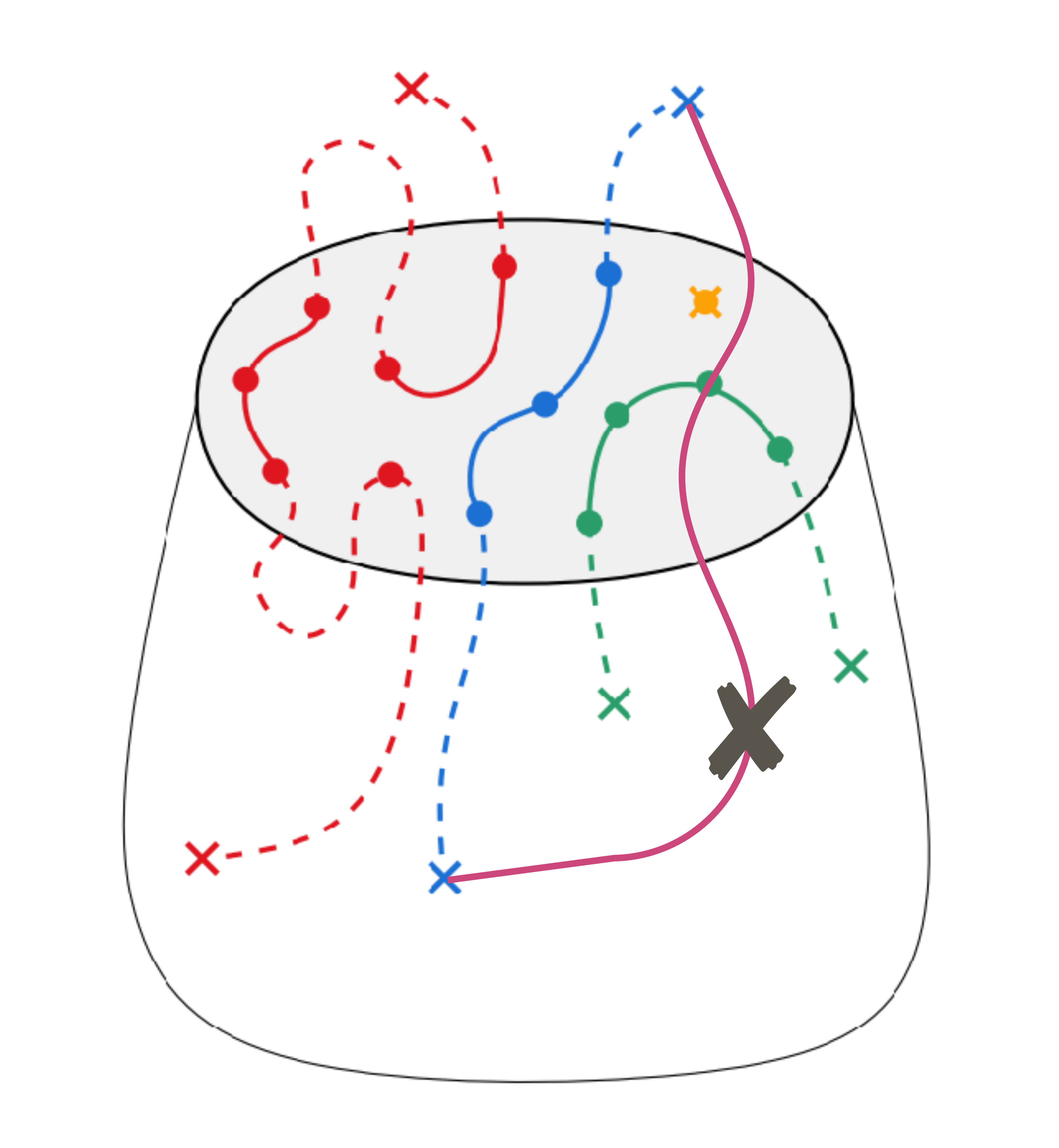

- Tracking endpoints is essential to ensure that the resulting paths remain isometric.
- Simple idea, until we start writing!
- We don't need actual endpoint but only its distance from all the vertices in the bag.
- There are diam^{tw} many distant distance profile with respect to any bag.

Theorem 1. Isometric Path Partition admits an algorithm running in time $n^{\mathcal{O}(\mathrm{tw})}$.

- Can we obtain FPT algorithm parameterized by treewidth?

Corollary~1. Isometric Path Partition admits an algorithm running in time $\mathrm{diam}^{\mathcal{O}(\mathrm{tw}^2)} \cdot n^{\mathcal{O}(1)}$.

- Can we obtain an improved FPT algorithm?

Theorem 1. Isometric Path Partition admits an algorithm running in time $n^{\mathcal{O}(\mathrm{tw})}$.

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Theorem 2. Isometric Path Partition is W[1]-hard when parameterized by the treewidth of the input graph.

Corollary~1. Isometric Path Partition admits an algorithm running in time $\mathrm{diam}^{\mathcal{O}(\mathrm{tw}^2)} \cdot n^{\mathcal{O}(1)}$.

- Can we obtain an improved FPT algorithm?

Theorem 3. Unless the Randomized-ETH fails, Isometric Path Partition does not admit an algorithm running in time $\mathrm{diam}^{o(\mathrm{tw}^2/\log^3(\mathrm{tw}))} \cdot n^{\mathcal{O}(1)}$.

Theorem 2. Isometric Path Partition is W[1]-hard when parameterized by the treewidth of the input graph.

- Resolves the open questions in Dumas et al. [SIDMA'24] and Fernau et al [TCS'25].
- Belmonte et al. [SIDMA'22] outline the list of problems that are W[1]-hard but XP parameterized by treewidth.
  Two themes:
  - problems involving weights, lists, or iterative processes,
  - metric-based problems like Metric Dimension, Geodetic Set, Distance d-Dom-Set, Distance d-Ind-Set, etc
- Metric property is critical as Path Partition or Hamiltonian Path are FPT parameterized by treewidth.

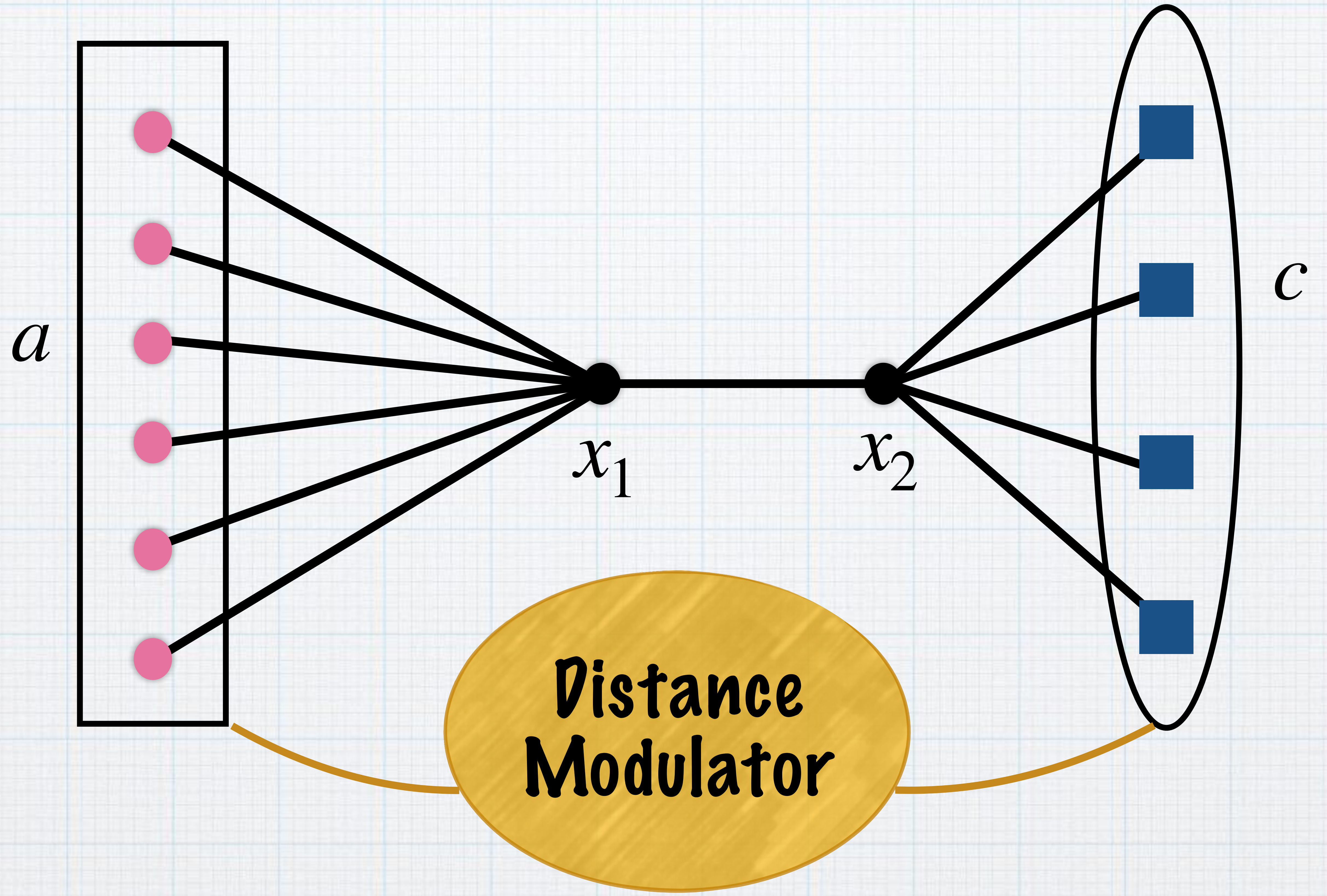
Theorem 3. Unless the Randomized-ETH fails, Isometric Path Partition does not admit an algorithm running in time $\mathrm{diam}^{o(\mathrm{tw}^2/\log^3(\mathrm{tw}))} \cdot n^{\mathcal{O}(1)}$.

- Randomized-ETH: There is no randomized algorithm that decides 3-SAT on instances with n variables in time $2^{o(n)}$ and succeeds with probability at least 2/3 on every input.
- These types of lower bounds and (almost) matching algorithms are relatively rare.

Brief overview of (ideas of) the reduction

Vertices corr. to assignments of some variables of 3-SAT

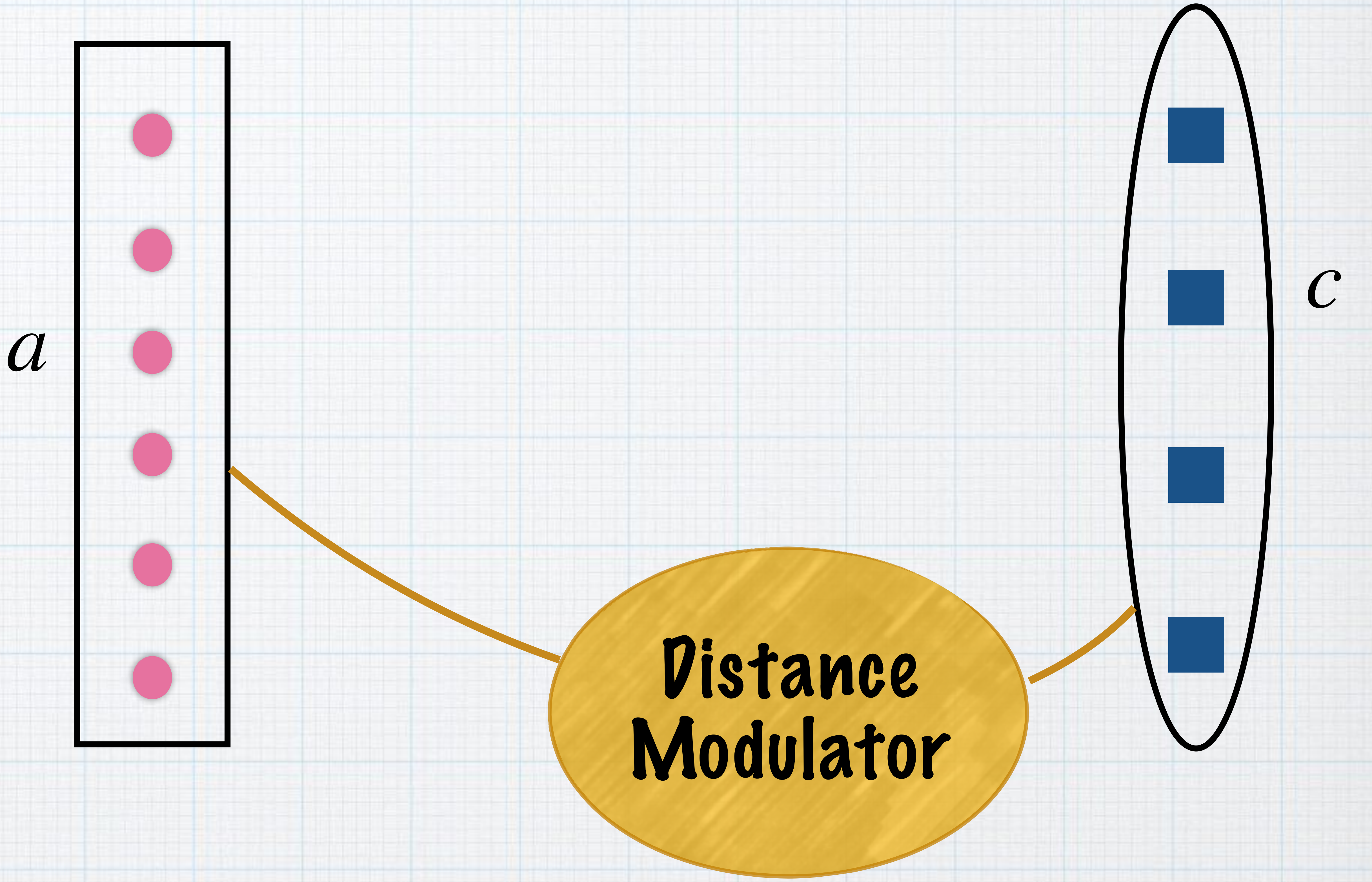
≡ possible starting point for isometric paths



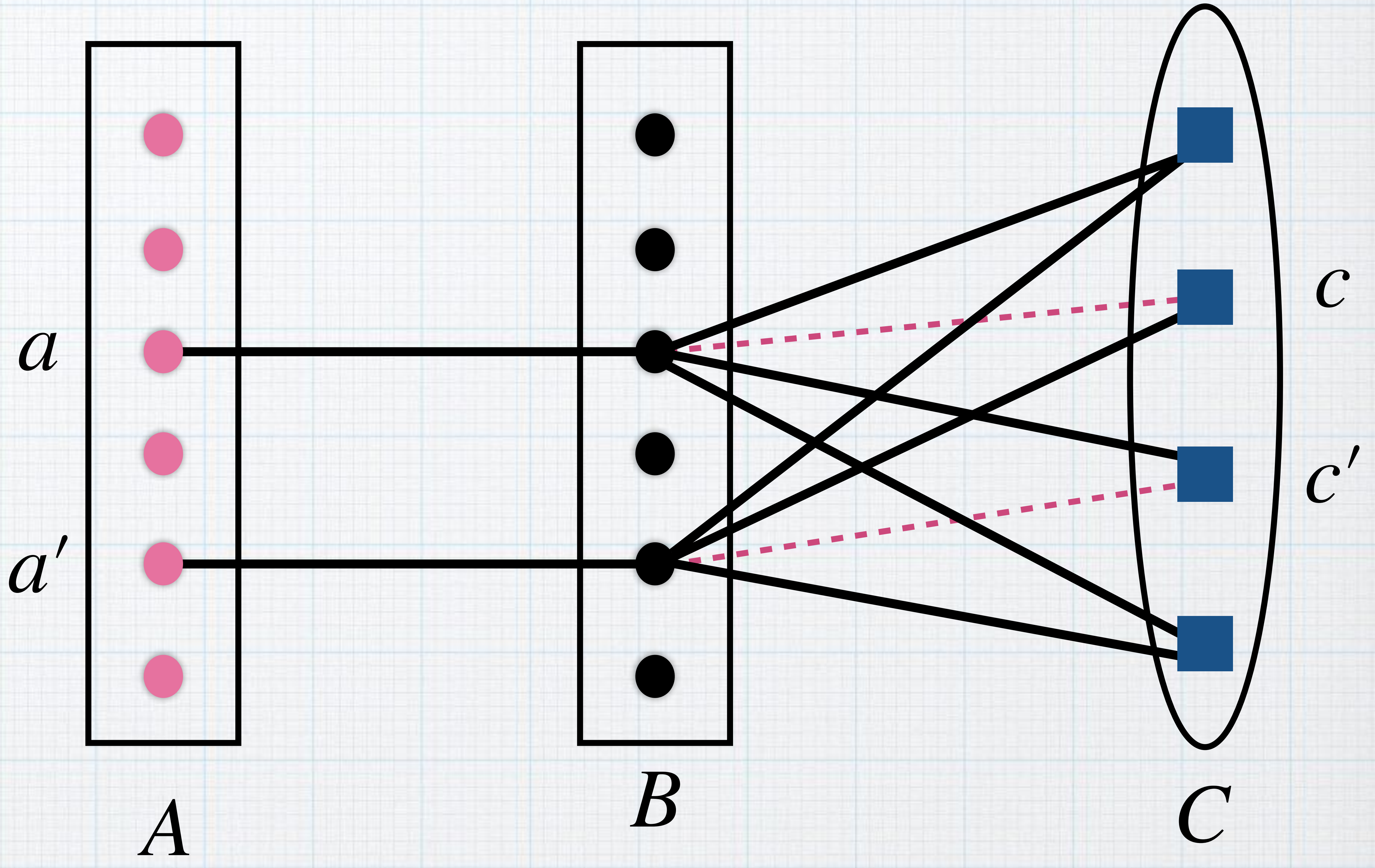
Vertices corr. to some clauses of 3-SAT

- Every assignment satisfies at most one clause in this group.
- Wishful thinking: Path starting at a can only be extended the unique clause c satisfied by the assignment corr. to a

- Given a function $f : A \mapsto C$ construct a graph G such that for any $a \in A$,
  - dist(a, f(a)) = 3 and
  - dist(a, c) $= 2$ for any $c \neq f(a)$.



- Given a function $f : A \mapsto C$ construct a graph G such that for any $a \in A$,
  - dist(a, f(a)) = 3 and
  - dist(a, c') $= 2$ for any $c' \neq f(a)$.



- Add clique B of size $|A|$.
  Add matching edges across A and B, and
  Add biclique minus `matching edges' (wrt f) across B and C.

- Can we reduce the size of B?

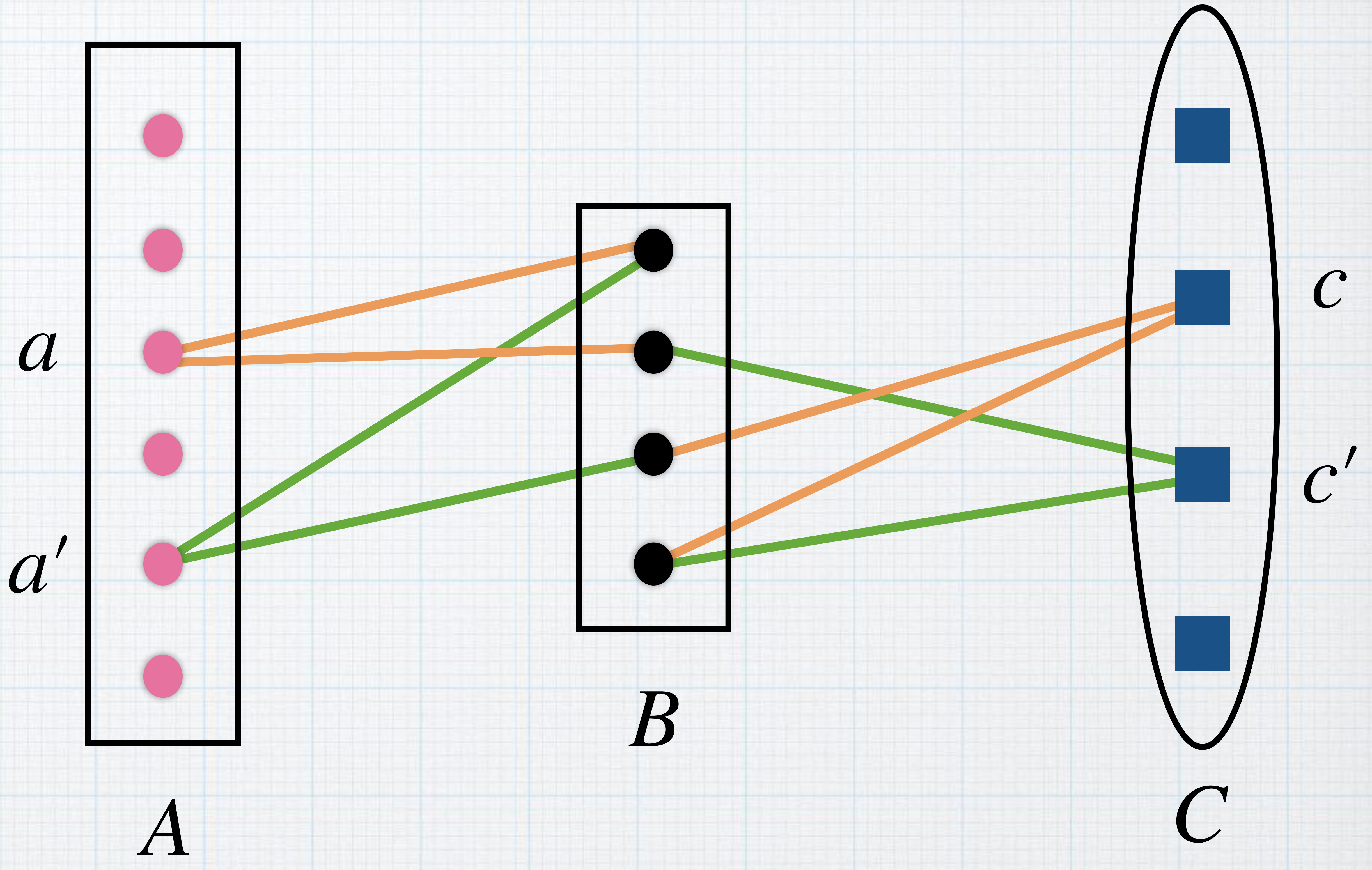
- Sperner Family: Collection $\mathscr{F}$ of subsets of a universe such that for any two sets X_1, X_2 in $\mathscr{F}$ neither $X_1 \subseteq X_2$ nor $X_2 \subseteq X_1$. i.e. $X_1 \cap \bar{X}_2 \neq \emptyset$ for any X_2 in $\mathscr{F} \setminus \{X_1\}$

- For example, consider ground set $\{1,2,\ldots,2p\}$ and define $\mathscr{F}$ as the collection subsets of size exactly p.

- For ground set of size $2p$, we get a Sperner family of size $2^{\mathcal{O}(p)}$.

  Alternately, to get Sperner family of size $|A|$, the ground set of size $\mathcal{O}(\log(|A|))$ is sufficient.

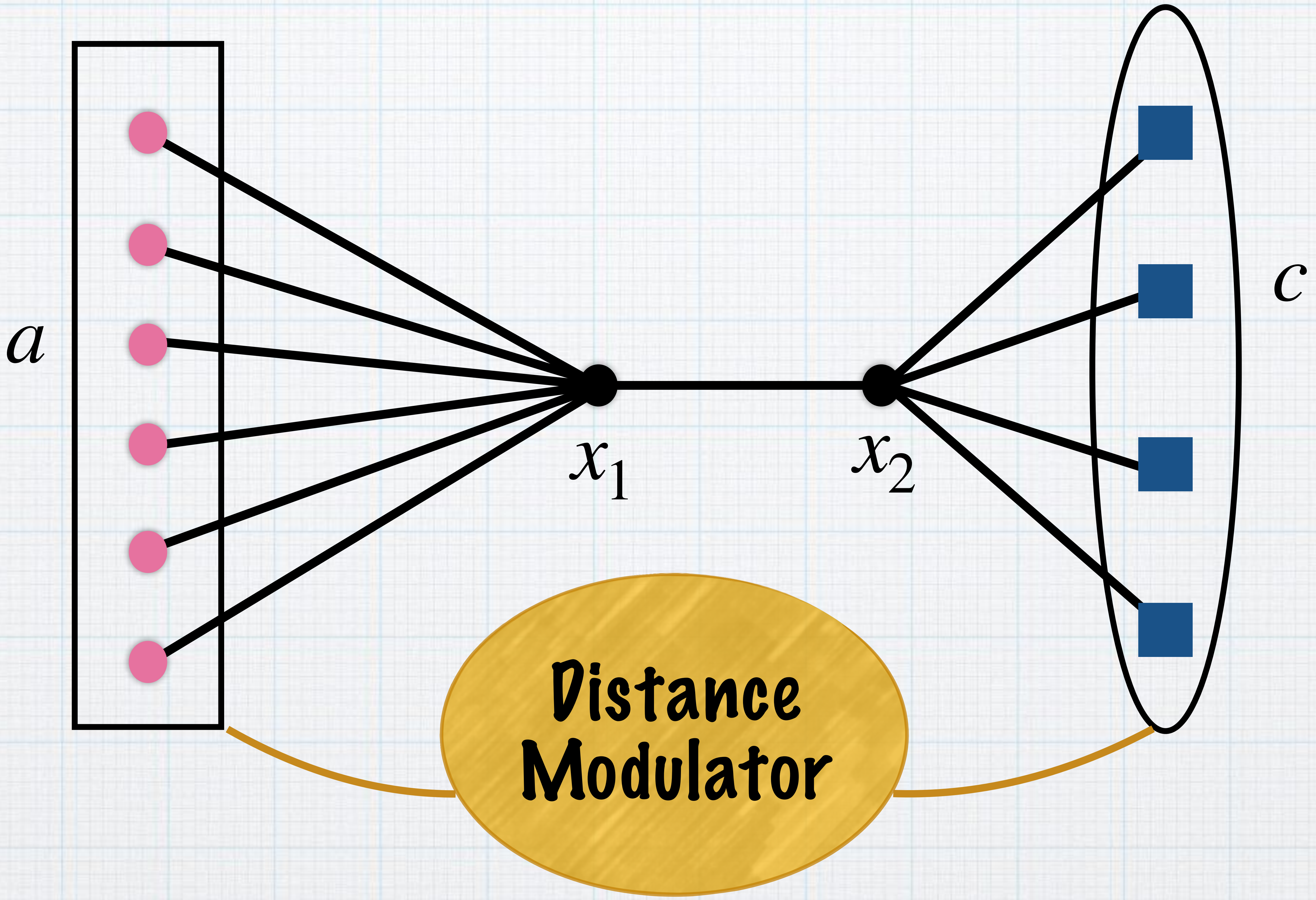
- Given a function $f : A \mapsto C$ construct a graph G such that for any $a \in A$,
  - dist(a, f(a)) = 3 and
  - dist(a, c') $= 2$ for any $c' \neq f(a)$.



- Add ground set B of size $\log(|A|)$, and make it a clique. Assign a Sperner set to each element in A. Suppose, rep$(a) \equiv \{1,2\}$ and rep$(a') \equiv \{1,3\}$. For every $c = f(a)$, assign c the complementary set of a.
- The graph has desired property, and $|B| \in \mathcal{O}(\log |A|)$.

Brief overview of (ideas of) the reduction

Vertices corr. to assignments of some variables of 3-SAT $\equiv$ possible starting point for isometric paths



Vertices corr. to some clause

- Every assignment satisfies at most one clause in this group.
- Path starting at a can only be extended the unique clause c satisfied by the corr. asst with distance modular $\mathcal{O}(\log |A|)$.

Sparse 3-SAT

Input: A 3-SAT formula with $n = p^2$ variables and n clauses.

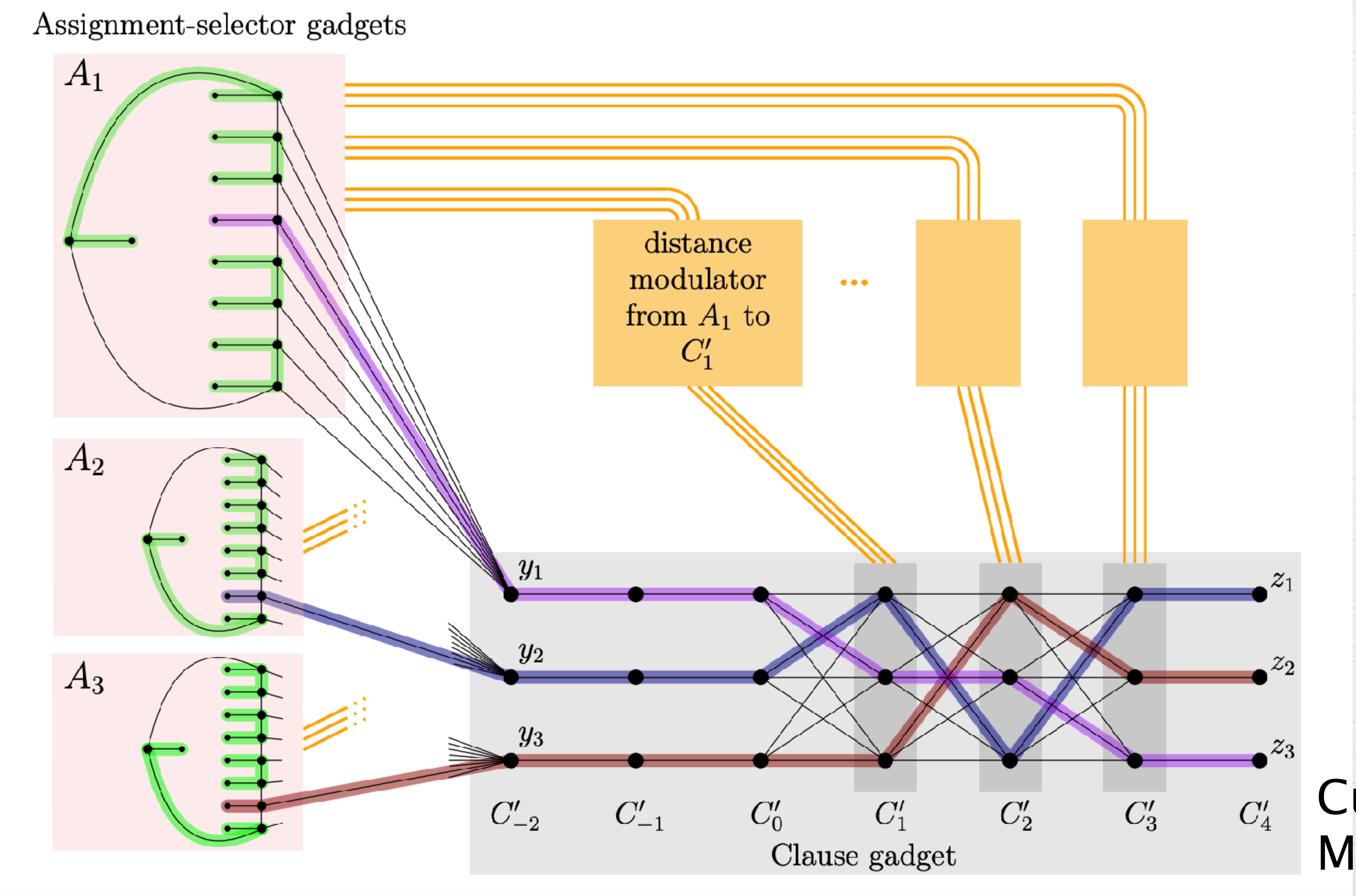
A partition $V_1, V_2, \ldots, V_p$ of variables.

A partition $C_1, C_2, \ldots, C_p$ of clauses.

The number of variables of V_i which appear in at least one clause of C_j is at most one.

Output: Is there a satisfying assignment?

Thm [GHLM, TCS'24]. Unless the Randomized ETH fails, Sparse 3-SAT does not admit an algorithm running in time $2^{o(n)}$.



Curtesy: M Meri

Open Questions

- Is Isometric Path Partition FPT when parameterized by the solution size k?
- Is Isometric Path Partition FPT when parameterized by treewidth on planar graphs?

  (The present reduction for W[1]-hardness on general graphs produces highly non-planar graph.)
- Similar question for other metric based problems Metric Dimension, Geodetic Set, Distance d-Dom-Set, Distance d-Ind-Set, etc.

Dank u! Merci! Danke!